# MATH 761: DIFFERENTIABLE MANIFOLDS (UW-MADISON, FALL 2024)

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# Part 1. Topological and smooth manifolds

## 1. Topological manifolds: definition and examples (Wed 9/4)

1.1. **The definition.** We want to study a class of topological spaces that retains certain essential features of $\mathbb{R}^n$. Our first attempt is the following.

**Definition 1.1.** Let $n \in \mathbb{N}$. A topological space $M$ is called a **topological manifold** of dimension $n$ if it is:

1. **Hausdorff**: given any $p \neq q \in M$, there exist open neighborhoods $U \ni p$ and $V \ni q$ such that $U \cap V = \varnothing$.
2. **Second-countable**: $M$ has a countable basis. (A basis $\mathcal{B}$ is a collection of open sets such that any open set is a union of elements of $\mathcal{B}$.)
3. **Locally Euclidean**: For each $p \in M$, there exists a neighborhood $U \ni p$, an open set $\hat{U} \subset \mathbb{R}^n$, and a homeomorphism
$$\varphi : U \xrightarrow{\sim} \hat{U}.$$

With this definition comes the following terminology.

- $\varphi$ is called a *coordinate chart*.
- If $\varphi(p) = 0$, we say that the chart is *centered at p*.
- If $\hat{U} = B_r(0)$ for some $r > 0$ and $\varphi(p) = 0$, the chart is called a *coordinate ball* centered at $p$.
- If we write
$$\varphi(q) = \left(x^1(q), \ldots, x^n(q)\right),$$
for $q \in U$, the functions $x^1, \ldots, x^n$ are called *local coordinates*.

1.2. **Examples.** We will spend the rest of class today giving examples of manifolds.

**Example 1.2.** Nature is full of manifolds. For instance, the surface of the Earth is a manifold which people assumed to be $\mathbb{R}^2$ for most of human history. In fact it turned out to be only *locally* Euclidean, but globally to have the shape of the 2-sphere. We still do not know the shape of the universe (seen as a manifold of dimension at least four).

For the rest of the semester we will limit ourselves to examples of a purely mathematical nature.

**Example 1.3.** Let $\Omega \subset \mathbb{R}^n$ be any open set, with the subspace topology. It is Hausdorff and second-countable because $\mathbb{R}^n$ is. It is also locally Euclidean, with a single coordinate chart: $U = \hat{U} = \Omega, \varphi = \mathrm{Id}$.

**Example 1.4.** Let $V \subset \mathbb{R}^n$ be an open set and $f : V \to \mathbb{R}^k$ a continuous function. Consider the graph

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \mid y = f(x)\},$$

with the subspace topology. This is again Hausdorff and second-countable. To make a coordinate chart, let $U = \Gamma(f)$, $\hat{U} = V \subset \mathbb{R}^n$, and consider the restriction of the projection to the first factor:

$$\begin{aligned} \varphi = \pi_1 : \Gamma(f) &\to V \\ (x, y) &\mapsto x. \end{aligned} \tag{1.1}$$

This has an inverse function given by

$$x \mapsto (x, f(x)),$$

which is continuous since it is the product of two continuous functions. Therefore $\varphi$ is a homeomorphism, as required.

**Example 1.5.** Let

$$S^n = \{x \in \mathbb{R}^{n+1} \mid (x^1)^2 + \cdots + (x^{n+1})^2 = 1\},$$

with the subspace topology. This is called the $n$-sphere, and we will see that it is a manifold of dimension $n$. As above it is Hausdorff and 2nd-countable since it is a subspace of $\mathbb{R}^{n+1}$. We will now describe coordinate charts that cover $S^n$.

Fix $i \in \{1, \ldots, n+1\}$, and let

$$U_i^+ = \{x \in S^n \mid x^i > 0\}$$

$$U_i^- = \{x \in S^n \mid x^i < 0\}.$$

Also let $\hat{U}_i = B_1(0) \subset \mathbb{R}^n$. Define the coordinate charts

$$\begin{aligned} \varphi_i^\pm = \pi_i : U_i^\pm &\to B_1(0) \\ (x^1, \ldots, x^{n+1}) &\mapsto (x^1, \ldots, \hat{x}^i, \ldots, x^{n+1}). \end{aligned}$$

(The notation $\hat{x}^i$ means that we actually *skip* the $i$'th entry on the list. Another way to write it would be: $(x^1, \ldots, \hat{x}^i, \ldots, x^{n+1}) = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1})$.)

To see that $\varphi_i^\pm$ is a homeomorphism, it suffices to write down the following inverse function.

$$(u^1, \ldots, u^n) \mapsto (u^1, \ldots, u^{i-1}, \pm\sqrt{1 - \sum_i (u^i)^2}, u^{i+1}, \ldots, u^n).$$

It is easy to check that this is indeed an inverse function of $\varphi_i^\pm$.

We can use the collection of $2(n+1)$ charts $\{U_i^\pm\}$ (this is called an *atlas*, as we'll discuss next week) to show that $S^n$ is locally Euclidean, as follows. Given a point $x \in S^n$, we must have $x^i \neq 0$ for some $i$. If $x^i > 0$ then $x \in U_i^+$, whereas if $x^i < 0$ then $x \in U_i^-$. Hence $x$ is contained in a coordinate chart. Since $x \in S^n$ was arbitrary, this shows that $S^n$ is locally Euclidean.

**Example 1.6.** Generalizing the previous example, we will prove later that the inverse image of any regular value of a $C^1$ function from $\mathbb{R}^n$ to $\mathbb{R}^m$, $n \geq m$, is a manifold of dimension $n-m$. This is the content of the *implicit function theorem*, which is the source of many very interesting examples of manifolds.

**Example 1.7.** The Cartesian product of two manifolds, $M \times N$, is again a manifold. The coordinate charts are given by products of the charts on $M$ and $N$ :

$$U \times V \xrightarrow{\varphi \times \psi} \mathbb{R}^m \times \mathbb{R}^n.$$

**Example 1.8.** The **2-torus** is defined to be the Cartesian product

$$\mathbb{T}^2 = S^1 \times S^1.$$

This manifold has many equivalent representations. You can draw it as a "doughnut" in Euclidean space. You can also define it as the unit square with boundary points identified as follows:

$$[0,1]^2 / (x,0) \sim (x,1), (0,y) \sim (1,y),$$

with the quotient topology. It is a simple exercise to write down a homeomorphism between this space and $\mathbb{T}^2 = S^1 \times S^1$. Finally, since $S^1 = \mathbb{R}/\mathbb{Z}$, where $n \in \mathbb{Z}$ acts by $x \mapsto x + n$, we have

$$\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}^2/\mathbb{Z}^2,$$

where $(m,n) \in \mathbb{Z}^2$ acts by

$$(x,y) \mapsto (x+m, y+n).$$

The map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}$ is the best example of a *covering map*, which is a concept that I hope you have seen before. If you are familiar with covering maps, you should have no trouble proving the following fact: *if $M \to N$ is a covering map and $M$ is a topological manifold, then $N$ is also a topological manifold.*

**Example 1.9.** Define the **Möbius strip:**

$$M = [0,1] \times \left(-\frac{1}{2}, \frac{1}{2}\right) \Big/ (0,y) \sim (1,-y),$$

with the quotient topology based on this equivalence relation. It is an exercise in the quotient topology (on your first homework) to show that $M$ is a topological manifold. Note that as with the torus, we have the equivalent description

$$M \cong \mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right) \Big/ \mathbb{Z},$$

where $n \in \mathbb{Z}$ acts by $(x,y) \mapsto (x+n, (-1)^n y)$.

**Example 1.10.** It is possible to close up the Möbius strip to obtain the *Klein bottle*:

$$K = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \Big/ \left(-\frac{1}{2}, y\right) \sim \left(\frac{1}{2}, -y\right), \left(x, -\frac{1}{2}\right) \sim \left(-x, \frac{1}{2}\right).$$

This a compact manifold that can be thought of as a "twisted" version of the 2-torus. It has the interesting property that it *cannot* be embedded in $\mathbb{R}^3$ without crossing itself. This requires a fair bit of algebraic topology to prove.

**Example 1.11.** As a set, the **real projective space** of dimension $n$ is defined to be

$$\mathbb{RP}^n = \{\text{1-dimensional subspaces of } \mathbb{R}^{n+1}\}.$$

We must also specify the topology on $\mathbb{RP}^n$. Letting

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$$
$$x \mapsto [x] = \{cx \mid x \in \mathbb{R}\}$$

be the canonical projection, we endow $\mathbb{RP}^n$ with the quotient topology induced by $\pi$.

We now describe a collection of coordinate charts covering $\mathbb{RP}^n$. Given $i \in \{1, \ldots, n+1\}$, let

$$V_i = \{x \in \mathbb{R}^{n+1} \mid x^i \neq 0\} \subset \mathbb{R}^{n+1}.$$

These are "saturated" open sets for the map $\pi$, i.e., if $x \in V_i$ then the whole fiber $\pi^{-1}([x]) \subset V_i$. By definition of the quotient topology,

$$U_i := \pi(V_i)$$

is open in $\mathbb{RP}^n$. To make $U_i$ into a coordinate chart, we let

$$\varphi_i : U_i \to \mathbb{R}^n =: \hat{U}_i$$
$$[x] = \left[x^1, \ldots, x^n\right] \mapsto \left(\frac{x^1}{x^i}, \ldots, \frac{\hat{x}^i}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right).$$

It is easy to check that $\varphi_i([cx]) = \varphi_i([x])$, so $\varphi_i$ is well-defined. Since $\varphi_i \circ \pi : V_i \to \mathbb{R}^n$ is continuous, by the universal property of the quotient topology, the map $\varphi_i$ is also continuous. To check that $\varphi_i$ is a homeomorphism, we need only write down the inverse map:

$$\left(u^1, \ldots, u^n\right) \mapsto \left[u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n\right], \tag{1.2}$$

which one easily checks is an inverse of $\varphi_i$. Therefore $\varphi_i$ is a coordinate chart, for each $i$. (Notice that the codomain $\hat{U}_i$ of each chart is equal to $\mathbb{R}^n$ itself! This is convenient.) Since $\cup_i U_i = \mathbb{RP}^n$, we have shown that $\mathbb{RP}^n$ is locally Euclidean.

To recap, we can write these charts in the following way:

$$U_i = \{\left[u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n\right] \mid (u^1, \ldots, u^n) \in \mathbb{R}^n\}.$$

This expression makes it almost obvious that these are coordinate charts covering $\mathbb{RP}^n$.

On your first homework you are asked to show that in addition to being locally Euclidean, $\mathbb{RP}^n$ is Hausdorff, second-countable, and *compact*; it is therefore a compact topological manifold. You can either show this directly using the coordinate charts that we just constructed, or you can make use of the following fact:

**Proposition 1.12.** $\mathbb{RP}^n \cong S^n / \pm 1$.

*Proof.* We will write down continuous maps between these two spaces which are inverses. In one direction, we can restrict the canonical projection to $S^n \subset \mathbb{R}^{n+1} \setminus \{0\}$ :

$$\pi|_{S^n} : S^n \to \mathbb{RP}^n.$$

Since this is the restriction of a continuous map to a subspace, it is again continuous. It is also surjective, since every 1-dimensional subspace of $\mathbb{R}^{n+1}$ contains unit vectors. In fact, given $x \in \mathbb{R}^{n+1} \setminus \{0\}$, we clearly have

$$[x] \cap S^n = \left\{ \pm \frac{x}{|x|} \right\}.$$

By the universal property of the quotient topology on $S^n / \pm 1$, the map $\pi|_{S^n}$ descends to a continuous map $S^n / \pm 1 \to \mathbb{RP}^n$ :

$$\begin{array}{ccc} S^n & \longrightarrow & \mathbb{R}^{n+1} \setminus \{0\} \\ \downarrow & \searrow^{\pi|_{S^n}} & \downarrow \pi \\ S^n / \pm 1 & \dashrightarrow & \mathbb{RP}^n \end{array}$$

On the other hand, the map $x \mapsto \frac{x}{|x|}$ also descends to give a continuous map $\mathbb{RP}^n \to S^n / \pm 1$:

$$\begin{array}{ccc} S^n & \xleftarrow{\frac{x}{|x|} \leftarrow x} & \mathbb{R}^{n+1} \setminus \{0\} \\ \downarrow & \swarrow & \downarrow \pi \\ S^n / \pm 1 & \dashleftarrow & \mathbb{RP}^n. \end{array}$$

These maps are easily seen to be inverses. ☐

## 2. Properties of topological manifolds (Fri 9/6 - Mon 9/9)

Let $M$ be a topological manifold. Today we will discuss: what does being a manifold imply about $M$, as a topological space? We will see that manifolds behave in a more intuitive way than general topological spaces.

2.1. **Basis of precompact coordinate balls.** The properties we will describe all follow from the proposition below. The key point is the following.

**Definition/Lemma 2.1.** *Let $(U, \varphi)$ be a coordinate chart in $M$ and $V \subset U$ an open subset. Suppose that the closure of $\varphi(V)$ in $\mathbb{R}^n$ is compact and contained in $\hat{U}$. Then the closure of $V$ in $M$ is also compact and contained in $U$. In this case we write $V \Subset U$ and say that $V$ is* **compactly contained in** *$U$.*

*Proof.* Let

$$\tilde{V} = \varphi^{-1}\left(\overline{\varphi(V)}\right),$$

where $\overline{\varphi(V)}$ is the closure in $\mathbb{R}^n$. By assumption, $\overline{\varphi(V)} \subset U$ is compact, and since $\varphi^{-1}$ is continuous, $\tilde{V}$ is also compact. But $M$ is Hausdorff, so compact implies closed (exercise!). Therefore $\tilde{V}$ is also closed in $M$. Letting $\overline{V}$ be the closure of $V$ in $M$, we have

$$\overline{V} \subset \tilde{V} \subset \overline{V} \cap U \subset \overline{V},$$

so $\overline{V} = \tilde{V}$, which is compact and contained in $U$, as desired. ☐

**Proposition 2.2.** *$M$ has a countable basis consisting of precompact coordinate balls. (A set is called precompact if its closure is compact.)*

*Proof.* Since $M$ is second-countable, we can cover $M$ with countably many coordinate charts $\{(U_i, \varphi_i)\}_{i=1}^{\infty}$. Since $U_i \cong \hat{U}_i \subset \mathbb{R}^n$, we can choose a countable basis for the topology of $U_i$ consisting of coordinate balls $V_{i,j}$ such that $\varphi_i(V_{i,j}) \Subset \hat{U}_i$. By the previous Lemma, we also have $V_{i,j} \Subset U_i$.

We can now take $\{V_{i,j}\}_{i,j=1}^{\infty}$ as our countable precompact basis. □

2.2. **Connected components.** Next, recall that there are two notions of connectivity for a topological space. A space $X$ is **connected** if there do not exist nonempty subsets $A$ and $B$ such that $A \cup B = X$ but $A \cap \bar{B} = \bar{A} \cap B = \varnothing$. A space $X$ is **path-connected** if for any $p, q \in X$ there exists a continuous path $\gamma : [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

In general, path-connected implies connected: since the unit interval is connected, its image is also connected, so cannot be separated by sets $A$ and $B$ as in the definition of connectedness. However, the converse is not true in general.

When a space is not connected, one can still divide it up into **connected components:** $X_0 \subset X$ is a connected component if it is a maximal connected subset. One can check that given any point $x \in X$ and any two connected subsets containing $x$, their union is again connected; so every point is contained in a unique connected component. In particular, we can partition any topological space $X$ into a disjoint union of connected components:

$$X = \bigsqcup_{\alpha} X_{\alpha}.$$

One can also partition a space $X$ into **path components**, which are maximal path-connected subsets. This is easier, because being connected by a path is clearly an equivalence relation so gives a partition of the space.

**Proposition 2.3.** *The path components of $M$ are the same as the connected components. In particular, any manifold is a countable disjoint union of closed/open, path-connected subsets.*

*Proof.* It suffices to show that the path components of $M$ are all open, for then these give a partition into closed/open sets which must agree with the connected components.

Let $M_0$ be a path component of $M$ and $x \in M_0$. There exists a coordinate ball $B \ni x$. Since $B$ is path-connected, we have $x \sim y$ for all $y \in B$, therefore $B \subset M_0$. This shows that $M_0$ is open.

Since the components are open and $M$ is second-countable, there can only be countably many of them. □

For this reason, it usually causes no loss of generality to work with *connected* manifolds—this is sometimes included in the definition, for convenience.

### 2.3. **Paracompactness.**

**Proposition 2.4.** *$M$ is locally compact, i.e., every point is contained in a neighborhood that is contained in a compact subset.*

*Proof.* By Proposition 2.2, each point is in fact contained in a precompact open set. □

Next, we will make a definition that you may not have seen before, although it is very important.

Given two open covers $\mathcal{U}$ and $\mathcal{V}$ of a topological space $X$, we say that $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. Intuitively, the open sets of $\mathcal{V}$ are "smaller" than that of $\mathcal{U}$.

An open cover $\mathcal{U}$ is called **locally finite** if for all $x \in X$, there exists a neighborhood $W \ni x$ such that $W \cap U \neq \varnothing$ for only finitely many $U \in \mathcal{U}$.

**Definition 2.5.** A space $X$ is called **paracompact** if every open cover admits a locally finite refinement. (Note: the refinement need *not* be a subcover of the original open cover.)

**Theorem 2.6.** *A topological manifold $M$ is paracompact.*

The proof will be based on:

**Lemma 2.7.** *$M$ admits an* **exhaustion** *by compact subsets*

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

*such that $\cup_{i=1}^{\infty} K_i = M$ and $K_i \subset K_{i+1}^{\circ}$.*

*Proof.* Let $U_1, U_2, \ldots$ be a countable cover by precompact open sets.

Take $K_1 = \bar{U}_1$.

By compactness, $K_1$ is covered by the open sets $U_1, U_2, \ldots, U_{n_2}$ for some $n_2 \in \mathbb{N}$. Take $K_2 = \cup_{i=1}^{n_2} \bar{U}_i$, which is a finite union of compact sets, hence compact. We also have

$$K_2^{\circ} = \cup_{i=1}^{n_2} U_i \subset K_1,$$

as required.

Next, cover $K_2$ by $U_1, \ldots, U_{n_3}$, with $n_3 \geq n_2$, etc.

In this way, we can obtain the required exhaustion. □

*Proof of Theorem 2.6.* Given an open cover $\mathcal{U}$ of $M$, we must construct a locally finite refinement. Let $\mathcal{B}$ be any basis, and fix an exhaustion $K_i$, $i = 1, 2, \ldots$ as guaranteed by the lemma. Let

$$C_i = K_{i+1} \smallsetminus K_i^{\circ},$$

which is a closed subset of a compact set, hence compact. Let

$$W_i = K_{i+2}^{\circ} \smallsetminus K_{i-1},$$

which is open. We have $C_i \subset W_i$.

Now, for each $x \in C_i$, choose $B_x \in \mathcal{B}$ such that $B_x \ni x$ and $B_x \subset W_i \cap U$ for some $U \in \mathcal{U}$. Since $C_i$ is compact, we can cover it by finitely many such neighborhoods, which we label $B_{i,j}$, for $j$ in a finite range. Then $\{B_{i,j}\}_{j=1}^{j_{max}}$ covers $C_i$ and refines $\mathcal{U}$.

We now take the collection of all $B_{i,j}$ as our refinement of $\mathcal{U}$. To see that it is locally finite, let $x \in M$ and choose $k$ such that $x \in W_k$. Since $B_{i,j} \cap W_k \neq \varnothing$ only possibly for $i-2 \leq k \leq i+2$, we have the required local finiteness property. □

One consequence of paracompactness is that $M$ is *metrizable*. This abstract fact is not particularly useful, however. Later we will study a specific type of metric on a manifold, called a *Riemannian* metric, that is extremely useful.

Last, we mention another property of manifolds which is not essential for the class, but good to know about:

**Proposition 2.8.** *The fundamental group of a topological manifold is countable. The fundamental group of a compact topological manifold is finitely generated.*

*Proof.* See Lee, Prop. 1.16 for the first statement. The finite generation statement can be proven similarly. □

## 3. Smooth manifolds: definition and examples (Mon 9/9-Fri 9/13)

3.1. **The definition.** As stated above, we want to work with a class of spaces that retain the essential features of $\mathbb{R}^n$. We defined topological manifolds to be spaces with locally Euclidean topology, plus some extra reasonableness conditions. On such a space, the notion of a real-valued continuous function makes sense:

$$f : M \to \mathbb{R}$$

is continuous if it is continuous as a map between topological spaces.

However, we really want to work with spaces that retain the most essential feature of $\mathbb{R}^n$, namely, calculus. Topological manifolds are actually inadequate for this purpose, as we now explain.

To begin to do calculus, one needs a notion of differentiable function. This should be a function which, when precomposed with a coordinate chart

$$f \circ \varphi^{-1} : \hat{U} \subset \mathbb{R}^n \to \mathbb{R},$$

is differentiable. However, on a topological manifold, this might be true in one chart but not in another. For example, we have the following two charts on $M = \mathbb{R} = U = \hat{U}$.

$$\varphi(x) = x, \qquad \psi(u) = u^3.$$

Consider $f(x) = x$. Then $(f \circ \varphi^{-1})(x) = x$ is differentiable, whereas $(f \circ \psi^{-1})(u) = u^{1/3}$ is not. So the notion of a differentiable function on a topological manifold makes no sense in general.

To avoid this problem, we need to introduce the following definition, which is key for the course.

**Definition 3.1.** Let $k \in \{0, 1, \ldots\} \cup \{\infty\}$. A $C^k$ **atlas** for $M$ is a collection $\{(U_\alpha, \varphi_\alpha)\}$ of charts, covering $M$, such that for each $\alpha, \beta$, the **transition map**

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta\left(U_\alpha \cap U_\beta\right) \to \varphi_\alpha\left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n$$

is of class $C^k$, i.e., is $k$ times continuously differentiable.

Two $C^k$ atlases are said to be **($C^k$) equivalent** if their union is again a $C^k$ atlas.

A $C^k$ **structure** is an equivalence class of $C^k$ atlases on $M$.

A $C^k$ **manifold** is a Hausdorff, second-countable topological space $M$ equipped with a $C^k$ structure.

A $C^\infty$ manifold is also called a **smooth** or **differentiable manifold**.

**Exercise.** Show that a topological manifold is the same thing as a $C^0$ manifold.

**Definition 3.2.** Let $M$ be a smooth manifold and $V \subset M$ an open set. A function $f : V \to \mathbb{R}$ is said to be **smooth** if for each $x \in V$ there exists a coordinate chart $(U, \varphi)$ such that

$$f \circ \varphi^{-1} : \varphi(V \cap U) \subset \mathbb{R}^n \to \mathbb{R}$$

is smooth.

Let $M$ and $N$ be smooth manifolds. A continuous map $f : M \to N$ is said to be **smooth** if for each $x \in M$ there exist compatible coordinate charts $(V, \varphi)$ and $(W, \psi)$ containing $x$ and $f(x)$, respectively, such that $f(V) \subset W$ and

$$\psi \circ f \circ \varphi^{-1} : \hat{V} \to \hat{W}$$

is smooth.

A smooth map $f$ is said to be a **diffeomorphism** if it has a smooth inverse.

**Exercise.** Check that the definitions of smooth function and smooth map make sense on a smooth manifold.

3.2. **Examples.** We can go over all the previous examples of topological manifolds and see that the charts we described are in fact smoothly compatible, i.e., define a smooth structure. For this, it suffices to compute the transition maps and check that they are smooth.

**Example 3.3.** In the case of $\mathbb{RP}^n$, to show that the charts $\{(U_i, \varphi_i)\}_{i=1}^{n+1}$ given above define a smooth structure, one can compute the transition function for $i > j$ :

$$\varphi_j \circ \varphi_i^{-1}(u^1, \dots, u^n) = \left( \frac{u^1}{u^j}, \dots, \frac{u^{j-1}}{u^j}, \frac{u^{j+1}}{u^j}, \dots, \frac{u^{i-1}}{u^j}, \frac{1}{u^j}, \frac{u^i}{u^j}, \dots, \frac{u^n}{u^j} \right).$$

Since $u^j > 0$ on $\varphi_i(U_i \cap U_j)$, this is smooth, as required.

**Example 3.4** (Stereographic charts)**.** Here is another way of defining the smooth structure on $S^n$, which goes back to Riemann.

Let $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$ be the North and South poles on $S^n$, respectively. Denote $V^+ = S^n \smallsetminus \{N\}$ and $V^- = S^n \smallsetminus \{S\}$. We can make $V^\pm$ into charts by taking the linear projection onto the equatorial plane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. A similar triangles calculation gives

$$X \mapsto x = \frac{1}{1 - X^{n+1}} \left( X^1, \dots, X^n \right), \qquad X \mapsto y = \frac{1}{1 + X^{n+1}} \left( X^1, \dots, X^n \right).$$

These are charts on $V^\pm$ with codomain $\mathbb{R}^n$. One can check that $N, X$, and $x$ are collinear, as are $S, X$, and $y$.

To show that this is a chart, we need to derive a formula for the inverse function:

$$x \mapsto X = \frac{1}{|x|^2 + 1} \left( 2x^1, \ldots, 2x^n, |x|^2 - 1 \right).$$

This is left as a homework exercise. We also need to compute the transition map. This turns out to be given just by inversion in the unit circle:

$$x \mapsto y = \frac{x}{|x|^2}.$$

Since $x \neq 0$ on $V^+ \cap V^-$, the transition map is smooth. The stereographic charts therefore define a smooth structure on $S^n$, which you can check is equivalent to the one defined by the hemisphere charts $U_i^{\pm}$ (exercise).

**Example 3.5.** Here is an example of a different flavor.

Let $V$ be a finite-dimensional real vector space. We first topologize $V$ using any norm; since any two norms are equivalent up to constants, this topology is independent of the norm.

We now define a canonical atlas on $V$ as follows. Given any choice of basis $e = \{e_1, \ldots, e_n\}$ for $V$, we have a map

$$\begin{aligned} E_e : \mathbb{R}^n &\to V \\ (x^1, \ldots, x^n) &\mapsto \sum_{i=1}^n x^i e_i. \end{aligned}$$

Since $e$ is a basis, this map has a linear inverse, which is continuous. We take

$$\varphi_e = E_e^{-1} : V \to \mathbb{R}^n.$$

We claim that the collection of charts $\{(\mathbb{R}^n, \varphi_e)\}$, where $e$ runs over all possible bases, is a smooth atlas. Let $e, \tilde{e}$ be two bases, and write

$$e_i = \sum_j A^j{}_i \tilde{e}_j$$

for an invertible matrix $\left(A^i{}_j\right)$. For $x = \varphi_e^{-1}\left(x^1, \ldots, x^n\right) \in \mathbb{R}^n = \hat{U}_e$, we have

$$\begin{aligned} \varphi_{\tilde{e}} \circ \varphi_e^{-1}(x) = \varphi_{\tilde{e}} \circ E_e(x) &= \varphi_{\tilde{e}} \left( \sum_i x^i e_i \right) \\ &= \sum_i x^i \varphi_{\tilde{e}}(e_i) \\ &= \sum_{i,j} x^i A^j{}_i \varphi_{\tilde{e}}\left(\tilde{e}_j\right) \\ &= \left( \sum_i x^i A^1{}_i, \ldots, \sum_i x^i A^n{}_i \right) \in \mathbb{R}^n = \hat{U}_{\tilde{e}}. \end{aligned}$$

This is a smooth map, so we indeed have a smooth structure.

Needless to say, the notion of a smooth function coincides with the standard one on $\mathbb{R}^n$ after identifying $V \cong \mathbb{R}^n$ using any choice of basis.

**Example 3.6.** Here is another key example. Notice that the space of $m \times n$ real matrices, $\mathrm{Mat}_{\mathbb{R}}^{m\times n}$, is identical with $\mathbb{R}^{m\cdot n}$, so can be considered a Euclidean space. Define

$$\mathrm{GL}(n,\mathbb{R}) = \{X \in \mathrm{Mat}_{\mathbb{R}}^{n\times n} \mid X \text{ is invertible}\}.$$

We know of course that $X$ is invertible iff $\det X \neq 0$. Since $\det X$ is a polynomial in the matrix coefficients, the subset $\{\det X = 0\}$ is closed. Hence $\mathrm{GL}(n,\mathbb{R})$ is open in $\mathrm{Mat}_{\mathbb{R}}^{n\times n}$; by Example 1.3, it is a manifold (indeed a smooth one, defined by a single chart).

Meanwhile, since the product of two invertible matrices is invertible, $\mathrm{GL}(n,\mathbb{R})$ is also a *group*. The group law (matrix multiplication) is given by a bilinear function of the matrix coefficients, so is smooth. By Cramer's formula, the inverse map $X \mapsto X^{-1}$ is also smooth. A smooth manifold with smooth group operations is called a **Lie group**, and these play a special role in this subject.

3.3. **Topological versus smooth manifolds.** It has been (and still remains) a celebrated problem to compare the category of topological manifolds with the category of smooth manifolds. Here are some historical landmarks.

*H. Whitney '36*: For $k \geq 1$, every $C^k$ structure is $C^k$ equivalent to a smooth structure.

*E. Moise '52*: For $n < 4$, every topological manifold carries a unique smooth structure up to diffeomorphism.

*J. Milnor '56*: *Exotic* smooth structures exist. In particular, the topological manifold $S^7$ carries smooth structures that are not diffeomorphic to the standard one.

*M. Kervaire '60*: There exists a compact 10-dimensional topological manifold that carries no smooth structure at all.

*M. Freedman / S. Donaldson '82*: $\mathbb{R}^4$ carries exotic smooth structures.

## 4. More smooth manifolds (Fri 9/13-Mon 9/16)

4.1. **Constructing a manifold from charts.** In the previous examples we always started with a given topological space and then cooked up an atlas of charts covering it. Of course, the atlas is the most important thing, and it is possible to go the other way: start with just a set, $M$, and a collection of maps satisfying certain conditions, and use this to define a topology on $M$ that makes it into a smooth manifold.

**Lemma 4.1** (Smooth manifold chart lemma, Lee 1.35)**.** *Let $M$ be a set and suppose we have a collection of subsets $U_\alpha$ covering $M$ together with maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ satisfying the following properties:*

1. *For each $\alpha$, $\varphi_\alpha$ is a bijection between $U_\alpha$ and $\varphi_\alpha(U_\alpha)$, which is an open subset of $\mathbb{R}^n$.*
2. *For each $\alpha$ and $\beta$, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.*
3. *Whenever $U_\alpha \cap U_\beta \neq 0$, the map*

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

*is smooth.*

4. *Countably many of the $U_\alpha$ cover $M$.*
5. *Whenever $p, q$ are distinct points in $M$, either there exists some $U_\alpha$ containing both $p$ and $q$ or there exist disjoint sets $U_\alpha$ and $U_\beta$ with $p \in U_\alpha$ and $q \in U_\beta$.*

*Then $M$ has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a chart.*

*Proof.* As a basis for the topology on $M$, we take the collection of all inverse images $\varphi_\alpha^{-1}(V)$ where $V$ is open in $\mathbb{R}^n$. To check that this is the basis for a topology, one needs to know that the intersection of two such sets, $\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$ is covered by basis elements. In fact, it is itself a basis element. For, since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth, it is continuous, and

$$Z = \left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^{-1}(W) \subset \varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

is open. It follows that

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}(V \cap Z)$$

is also a basis element, as claimed. Therefore these open sets form the basis for a topology.

It is easy to see that $\varphi_\alpha$ is a homeomorphism with this definition, and is clearly a smooth atlas. The Hausdorff property is clear from the last assumption: in the first case, we can separate the points by balls contained in $U_\alpha$, while in the second case, $U_\alpha$ and $U_\beta$ are the separating neighborhoods. Finally, the second-countability follows because each $U_\alpha$ is second-countable and by assumption we can take a countable subcollection that still covers $M$.

The uniqueness is also clear since the basis we described will be a basis for any topology in which the above requirements are satisfied. □

**Remark 4.2.** One can go even further and define the set $M$ by gluing the sets $U_\alpha$ using the equivalence relation coming from a given collection of transition maps, subject to the appropriate conditions. Some people really think of a smooth manifold as "a bunch of domains in Euclidean space glued together." In my opinion it's better to think of it as a preexisting set endowed with additional structure.

## 4.2. $\mathbb{CP}^n$ and Grassmannians.

We will now use the above lemma to do two more examples.

Let $K = \mathbb{R}$ or $\mathbb{C}$. These are topological fields on $\mathbb{R}$ and $\mathbb{R}^2$, respectively, so the vector space $K^n$ is identical to $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, respectively, and the field operations are all given by smooth maps. For instance, scalar multiplication gives a smooth map $K \times K^n \to K^n$, which restricts to a diffeomorphism $\{c\} \times K^n \to K^n$, for $c \neq 0$. Addition by a fixed vector is also clearly a diffeomorphism.

**Example 4.3.** By direct analogy with $\mathbb{RP}^n$, let

$$\mathbb{CP}^n = \{\text{1-dimensional complex subspaces of } \mathbb{C}^{n+1}\}.$$

Rather than proceeding via the quotient topology as in the case of $\mathbb{RP}^n$, we will use the previous Lemma to show that this is a smooth manifold of real dimension $2n$. (These approaches of course yield the same space.)

It is convenient to use charts of a slightly more general kind than above. Let $\langle \cdot, \cdot \rangle$ denote the Hermitian inner product on $\mathbb{C}^{n+1}$ which is $\mathbb{C}$-linear in the first factor and conjugate-linear in the second factor. Given a unit vector $w \in \mathbb{C}^{n+1}$, let

$$U_w = \{\ell \in \mathbb{CP}^n \mid \ell \notin w^\perp\}.$$

Clearly, if $\ell \in U_w$ then $\langle v, w \rangle \neq 0$ for all nonzero $v \in \ell$.

Let $\hat{U}_w = w^\perp$. This is an $n$-dimensional complex subspace of $\mathbb{C}^{n+1}$, therefore a $2n$-dimensional real subspace, which can be identified with $\mathbb{R}^{2n}$ using any choice of basis. This will be the codomain of our coordinate chart.

We let

$$\varphi_w([v]) = \frac{v}{\langle v, w \rangle} - w,$$

which is a well-defined map. The inverse is given by

$$x \mapsto [x + w].$$

So $\varphi_w$ is a bijection with $\mathbb{C}^n$, which is of course open, so (1) is satisfied. Given $w, w' \in \mathbb{C}^{n+1}$, for $x \in \varphi_w\left(\hat{U}_w \cap \hat{U}_{w'}\right)$, we have $\varphi_w^{-1}(x) \in U_{w'}$, so $\langle x + w, w' \rangle \neq 0$. This is an open condition, so (2) is satisfied. The transition map

$$x \mapsto \frac{x + w}{\langle x + w, w' \rangle} - w'$$

is smooth, so (3) is satisfied.

Note that the charts $U_i$ are simply the charts $U_{e_i}$ for the coordinate basis vectors $e_i$. So the finite collection $\{U_{e_i}\}_{i=1}^{n+1}$ covers $\mathbb{CP}^n$, and (4) is satisfied.

Finally, to check condition (5) of Lemma 4.1, we note that for any two lines $\ell, \ell' \in \mathbb{CP}^n$, it is easy to construct a nonzero vector $w \in (\ell^\perp)^c \cap (\ell'^\perp)^c$, so that $\ell$ and $\ell'$ both lie in the chart $U_w$. This guarantees (5).

Here is a famous example of a diffeomorphism.

**Proposition 4.4.** $\mathbb{CP}^1 \cong S^2$.

*Proof.* We can write down the following maps:

$$\begin{aligned} S^2 \setminus N &\to U_2 \\ (x, y, z) &\mapsto [x + iy, 1 - z] \end{aligned}$$

and

$$\begin{aligned} S^2 \setminus S &\to U_1 \\ (x, y, z) &\mapsto [1 + z, x - iy]. \end{aligned}$$

These are clearly smooth, and when composed with the coordinates on $U_1$ and $U_2$, agree with the stereographic projections up to a reflection; so it is clear that both are diffeomorphisms.

For $(x,y,z) \in S^2$ with $z \neq \pm 1$, we have

$$[x+iy, 1-z] = \left[(x+iy)(1+z), 1-z^2\right] = \left[1+z, \frac{x^2+y^2}{x+iy}\right] = [1+z, x-iy].$$

Therefore the two maps above coincide on $S^2 \setminus \{N,S\}$, giving a well-defined global diffeomorphism. ☐

**Remark 4.5.** For $n > 1$, $\mathbb{CP}^n$ is not a sphere. Intuitively, this is because

$$S^{2n} = \mathbb{R}^{2n} \sqcup \{pt\},$$

whereas

$$\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{CP}^{n-1},$$

so these are different ways of compactifying Euclidean space. Real projective space is another compactification. The following example gives yet another.

**Example 4.6.** Generalizing the previous example, for $K = \mathbb{R}$ or $\mathbb{C}$, we let

$$\mathrm{Gr}_k(K^n) = \{k\text{-dimensional linear subspaces of } K^n\}.$$

We shall describe a family of charts making this into a smooth manifold of dimension $k(n-k)$ in the real case and $2k(n-k)$ in the complex case.

Given any $(n-k)$-dimensional subspace $Q \subset K^n$, let[1]

$$U_Q = \{X \in \mathrm{Gr}_k(K^n) \mid X \cap Q = \{0\}\}.$$

We can make $U_Q$ into a coordinate chart as follows. Fix any $P \in U_Q$ and observe that $K^n = P \oplus Q$. After changing basis, we may assume that $P = K^k \times \{0\}$ and $Q = \{0\} \times K^{n-k}$ are coordinate planes.

Now, because $K^n = P \oplus Q$, any $X \in U_Q$ is the graph of a linear map from $P$ to $Q$. In particular, it is the span of a unique matrix $n \times k$ matrix of the form

$$\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ & A & \end{pmatrix}, \tag{4.1}$$

where $A$ is an $(n-k) \times k$ matrix. We take our chart to be

$$\varphi_Q : X \in U_Q \mapsto A \in \mathrm{Mat}_K^{(n-k)\times k}.$$

Since $X$ and $A$ determine each-other uniquely, this is bijective, so (1) is satisfied.

Now let $P', Q'$ be another pair of complementary subspaces. Note that $X \in U_{Q'} \cap U_Q$ iff the image of the projection from $X$ to $Q'$ has full rank. Since rank is determined by nonvanishing of minors, this is an open condition on the matrix (4.1), so (2) is satisfied.

To check smoothness of the transition map, note that by construction, the map $\varphi_Q^{-1}$ takes $A' \in \mathrm{Mat}_K^{(n-k)\times k}$ to the span of an $n \times k$ matrix which looks like (4.1) in a different basis.

[1] In Example 4.3, with this notation, we would have written $U_{w^\perp}$ instead of $U_w$.

Changing to the chosen basis for $P$ and $Q$, we obtain a matrix whose coefficients depend linearly on the entries of $A'$. We can write this matrix as

$$\begin{pmatrix} B \\ C \end{pmatrix},$$

where $B = B(A')$ is $k \times k$ and $C = C(A')$ is $(n-k) \times k$, and both depend smoothly (indeed, linearly) on $A'$.

Supposing that $X \in U_Q \cap U_{Q'}$, we must have $X \cap Q = 0$, so $X$ and $Q$ are complementary. This means that $B$ is invertible. We can multiply by $B$ on the right without changing the span, so

$$\varphi_{Q'}^{-1}(X) = \operatorname{Span}\begin{pmatrix} B \\ C \end{pmatrix} = \operatorname{Span}\begin{pmatrix} BB^{-1} \\ CB^{-1} \end{pmatrix} = \operatorname{Span}\begin{pmatrix} \mathbf{1} \\ CB^{-1} \end{pmatrix}.$$

We therefore have

$$\varphi_Q\left(\varphi_{Q'}^{-1}(A')\right) = CB^{-1} \in \operatorname{Mat}_K^{(n-k)\times k}.$$

This shows that the transition function is smooth on the overlap, so (3) is satisfied. Properties (4) and (5) can be established in a similar way to the previous example. We of course have $\operatorname{Gr}_1(K^n) = K\mathbb{P}^{n-1}$.

## 5. Smooth partitions of unity (Wed 9/18)

Today we will prove a basic lemma in the theory of (smooth) manifolds. Recall that the **support** of a real- or complex-valued function $f$ is defined by

$$\operatorname{supp} f = \overline{\{x \in M \mid f(x) \neq 0\}}.$$

**Definition 5.1.** Let $\mathcal{X} = \{X_\alpha\}$ be an open cover of a smooth manifold $M$. A smooth **partition of unity** subordinate to $\mathcal{X}$ is a collection of smooth functions $\psi_\alpha$ (indexed by the same set as $\mathcal{X}$) such that for each $\alpha$ :

(i) $0 \le \psi_\alpha \le 1$

(ii) $\operatorname{supp} \psi_\alpha \subset X_\alpha$

(iii) The collection $\{\operatorname{supp} \psi_\alpha\}$ is locally finite

(iv) $\sum_\alpha \psi_\alpha \equiv 1$.

Notice that item (iv) makes sense in view of item (iii). In other words, for each $x \in M$, the sum in (iv) has only finitely many nonzero terms, so is well-defined without any analysis.

**Definition 5.2.** Let $A \subset U \subset M$, with $U$ open and $\overline{A} \subset U$. A **bump function** $H$ for the pair $A \subset U$ is a continuous function on $M$ with:

- $0 \le H(x) \le 1$
- $H(x) \equiv 1$ for all $x \in A$

- $\operatorname{supp} H \subset U$.

**Lemma 5.3.** *Let $0 < r_1 < r_2 < \infty$. There exists a smooth bump function on $\mathbb{R}^n$ for the pair*

$$\overline{B_{r_1}(0)} \subset B_{r_2}(0).$$

*Proof.* Let

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \le 0. \end{cases}$$

For $t > 0$, we have $f'(t) = \frac{1}{t^2} e^{-\frac{1}{t}}$, which tends to zero as $t \searrow 0$, so $f'(t)$ is continuous on $\mathbb{R}$. Each higher derivative is again of the form $p\left(\frac{1}{t}\right) e^{-\frac{1}{t}}$, and so also tends to zero as $t \searrow 0$. The function $f(t)$ is therefore smooth on $\mathbb{R}$, with all derivatives vanishing on $(-\infty, 0]$, and with $0 < f(t) \le 1$ for $t > 0$.

Now define

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) - f(t - r_1)}.$$

One can check that this is a smooth bump function for $(-\infty, r_1] \subset (-\infty, r_2)$. Finally, for $x \in \mathbb{R}^n$, let

$$H(x) = h(|x|).$$

This is smooth for $|x| > 0$ by the chain rule, and since it is identically equal to one on a neighborhood of the origin, is also smooth there. □

We also need the following definition, which could have been made earlier but only becomes relevant now.

**Definition 5.4.** A **regular coordinate ball** $B \subset M$ is a coordinate ball centered at $p \in M$ for which there exists another coordinate ball $B'$ centered at $p$ with $B \Subset B' \Subset U \subset M$, where $U$ is the coordinate chart making these into coordinate balls. In other words, if $\varphi$ is the chart map for $U$, we have

$$\varphi(B) = B_{r_1}(0), \quad \varphi(B') = B_{r_2}(0), \quad r_1 < r_2,$$

with $B_{r_2}(0) \Subset \varphi(U)$. This guarantees that $\varphi(\bar{B}) = \overline{B_{r_1}(0)}$.

The proof of Proposition 2.2 in fact showed that we can always choose a countable basis for $M$ consisting of *regular* coordinate balls.

**Theorem 5.5.** *Suppose $M$ is a smooth manifold and $\mathcal{X} = \{X_\alpha\}$ is an open cover. There exists a smooth partition of unity $\{\psi_\alpha\}$ subordinate to $\mathcal{X}$.*

*Proof.* Recall that in the proof of paracompactness of $M$, Theorem 2.6, we in fact showed that a countable locally finite refinement of $\mathcal{X}$ can be chosen from any basis. Let $\{B_i\}_{i=1}^{\infty}$ be a countable locally finite refinement consisting of regular coordinate balls. It is easy to see from the definition of local finiteness (involving a neighborhood $W \ni x$) that the collection of closures $\{\bar{B}_i\}$ is again locally finite (using the same neighborhood). Since the balls are regular, there exists a collection $\{B_i'\}$ and chart maps $\varphi_i$ such that $\varphi_i(\bar{B}_i) = \overline{B_{r_1}(0)}$, and $\varphi_i(B_i') = B_{r_2}(0)$. For each $i$, let $H_i$ be the bump function on $\mathbb{R}^n$ guaranteed by (5.3). Let

$$f_i = \begin{cases} H_i \circ \varphi_i(x) & x \in B_i' \\ 0 & x \in M \setminus B_i'. \end{cases}$$

Let

$$f(x) = \sum_{i=1}^{\infty} f_i(x),$$

which is a locally finite sum, so gives a smooth positive function on $M$. Next, let

$$g_i(x) = \frac{f_i(x)}{f(x)}.$$

These are each smooth, bounded between zero and one, and clearly satisfy

$$\sum_i g_i \equiv 1.$$

Hence $\{g_i\}$ is a p.o.u. subordinate to $\{B_i\}$.

It remains to regroup and re-index to make the partition subordinate to the original cover $\mathcal{X}$. For each $i$, choose $a(i)$ such that $B_i \subset X_{a(i)}$. Define

$$\psi_\alpha = \sum_{i:a(i)=\alpha} g_i.$$

This clearly satisfies (i) and (iv) in the definition of p.o.u. Note that

$$\operatorname{supp} \psi_\alpha = \overline{\cup_{i:a(i)=\alpha} B_i}.$$

Since $\{\bar{B}_i\}$ is locally finite, one can pass the closure to the inside, giving

$$\operatorname{supp} \psi_\alpha = \cup_{i:a(i)=\alpha} \bar{B}_i \subset \mathcal{X}_\alpha,$$

which is (ii). Also, since each $g_i$ appears only once in $\psi_\alpha$, (iii) remains true. (Notice that many of the $\psi_\alpha$'s may be identically zero.) □

We can now do a few applications.

**Proposition 5.6.** *Let $A \subset U \subset M$ with $A$ closed and $U$ open. Then there exists a smooth bump function for the pair $A \subset U$.*

*Proof.* Take $U_0 = U, U_1 = M \setminus A$. Let $\{\psi_0, \psi_1\}$ be a p.o.u. subordinate to $\{U_0, U_1\}$. We have $\psi_1 \equiv 0$ on $A$, so $\psi_0 \equiv 1$ on $A$. Also, by definition, $\operatorname{supp} \psi_0 \subset U_0 = U$. Hence $\psi_0$ is the required bump function. □

**Remark 5.7.** It is possible to choose $\psi$ in the previous proposition such that $\psi^{-1}(1) = A$, i.e., the value 1 is taken on exactly on $A$ (whereas $\psi_0$ may be identically equal to one on a larger set). This follows from Lee, Theorem 2.29, which is also relevant to one of your homework problems.

**Definition 5.8.** An **exhaustion function** for $M$ is a continuous function on $M$ such that for each $c \in \mathbb{R}$,

$$f^{-1}(-\infty, c]$$

is compact. In particular, an exhaustion function gives a nice way to define an exhaustion by compact sets $K_n = f^{-1}(-\infty, n]$ in the sense of Lemma 2.7.

**Example 5.9.** $M = \mathbb{R}^n$, $f(x) = |x|^2$.

**Proposition 5.10.** *Every smooth manifold admits a positive, smooth exhaustion function.*

*Proof.* Let $\{U_i\}$ be any countable, locally finite cover by precompact open sets, and $\{\psi_i\}$ a subordinate p.o.u. Define

$$f(x) = \sum_{i=1}^{\infty} i\psi_i(x).$$

The sum is locally finite since the cover is. Moreover, we have

$$f^{-1}(-\infty, c] \subset \overline{\cup_{i=1}^{\lceil c \rceil} U_i} = \cup_{i=1}^{\lceil c \rceil} \bar{U}_i,$$

which is compact. The first inclusion is because if $x \notin U_i$ for $i = 1, \ldots, \lceil c \rceil$, then $x \in U_j$ for some $j > c$, which implies $f(x) \geq j > c$. ☐

# Part 2. The tangent space

## 6. Definition(s) of the tangent space (Fri 9/20)

Given any open subset $U \subset M$, we have constructed the ring of smooth functions $C^\infty(U)$. Also, given two smooth manifolds $M$ and $N$, we can talk about smooth maps $f : M \to N$ (ones for which the pullback of any local coordinate on $N$ is smooth on $M$). In particular, a **smooth path** in $M$ is just a smooth map

$$\gamma : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to M.$$

Supposing that $\gamma(0) = x$, we say that $\gamma$ is a smooth path through $x$. (One can equally well consider $C^1$ paths.)

**Question.** What space does the derivative "$\gamma'(0)$" belong to?

**Answer 1.** Given any coordinate chart $(U, \varphi)$ containing $x$, by shrinking $\varepsilon$ if necessary, we can assume that $\gamma(-\varepsilon, \varepsilon) \subset U$. We can then take the composition $\varphi \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$, and form the usual derivative

$$(\varphi \circ \gamma)'(0) = \lim_{t \to 0} \frac{\varphi(\gamma(t)) - \varphi(x)}{t}.$$

Note that we are using the vector-space structure of $\mathbb{R}^n$ to form the difference quotient, and the topology on $\mathbb{R}^n$ to take the limit. So, from a smooth path in $M$ and a coordinate chart, we get a vector in $\mathbb{R}^n$.

Given a different coordinate chart $(V, \psi)$, the corresponding vectors in $\mathbb{R}^n$ are related by the chain rule as follows:

$$\begin{aligned} (\psi \circ \gamma)'(0) &= \left(\psi \circ \varphi^{-1} \circ \varphi \circ \gamma\right)'(0) \\ &= d\left(\psi \circ \varphi^{-1}\right)_{\varphi(x)} (\varphi \circ \gamma)'(0), \end{aligned} \tag{6.1}$$

where $d\left(\psi \circ \varphi^{-1}\right)_{\varphi(x)}$ is the usual derivative of a map from $\mathbb{R}^n \to \mathbb{R}^n$. As one shows in multivariable calculus, the matrix corresponding to this linear map is the so-called *Jacobian*, which contains all the partial derivatives of the coordinate functions (see Example 6.1 below).

Now, to give the first definition of the tangent space at $x$, let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be any smooth atlas for $M$ (which one could take to be maximal for convenience). Let $A_x = \{\alpha \in A \mid U_\alpha \ni x\}$. Take one copy $\mathbb{R}^n_\alpha$ of $\mathbb{R}^n$ for each $\alpha \in A_x$. For any $v_\alpha \in \mathbb{R}^n_\alpha$ and $v_\beta \in \mathbb{R}^n_\beta$, declare $v_\alpha \sim v_\beta$ if and only if

$$d\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)_{\varphi_\alpha(x)} v_\alpha = v_\beta.$$

It's easy to check that this is an equivalence relation, so we can define

$$T_x^{(1)} M := \sqcup_{\alpha \in A_x} \mathbb{R}^n_\alpha / \sim .$$

For each $\alpha$, the map

$$v_\alpha \in \mathbb{R}^n_\alpha \mapsto [v_\alpha] \in T_x^{(1)} M$$

is clearly a bijection, since $[v_\alpha] \cap \mathbb{R}^n_\beta$ consists of one element for each $\beta \in A_x$. Moreover, since the derivative is a linear map, the identifications are all linear, so $T_x^{(1)} M$ inherits the same vector-space structure from each chart.

Given a path $\gamma(t)$ through $x$, we may let

$$\gamma'(0) = \left\{ \left( (\varphi_\alpha \circ \gamma)' (0) \right)_\alpha \mid \alpha \in A_x \right\} \in T_x^{(1)} M,$$

which is a well-defined equivalence class by (6.1).

**Example 6.1.** Let $\{x^j\}$ be a system of local coordinates on $U_\alpha$ and $\{y^i\}$ a system of local coordinates on $U_\beta$, both containing $x$. The transition map can be written as

$$\varphi_\beta \circ \varphi_\alpha^{-1} : (x^1, \dots, x^n)^T \mapsto \left( y^1(x^1, \dots, x^n), \dots, y^n(x^1, \dots, x^n) \right)^T.$$

The derivative of $\gamma$ is given in the $U_\beta$ coordinates by

$$\begin{aligned} \gamma'(0) &= \left[ \begin{pmatrix} \frac{dy^1(\gamma(t))}{dt} \\ \vdots \\ \frac{dy^n(\gamma(t))}{dt} \end{pmatrix}_\beta \right] \\ &= \left[ \begin{pmatrix} \sum_j \frac{dx^j(\gamma(t))}{dt} \frac{\partial y^1}{\partial x^j} \\ \vdots \\ \sum_j \frac{dx^j(\gamma(t))}{dt} \frac{\partial y^n}{\partial x^j} \end{pmatrix}_\beta \right], \end{aligned}$$

where we have applied the chain rule on each coordinate function. The vector on the inside is precisely the Jacobian matrix

$$\left( \frac{\partial y^i}{\partial x^j} \right)_{i,j=1}^n = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

multiplied by the derivative in the $U_\alpha$ chart,

$$\begin{pmatrix} \frac{dx^1(\gamma(t))}{dt} \\ \vdots \\ \frac{dx^n(\gamma(t))}{dt} \end{pmatrix}_\alpha.$$

This shows that that $d\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)_{\varphi_\alpha(x)}$ indeed acts by multiplication by the Jacobian matrix.

**Answer 2.** Fix a path $\gamma$ as above. Given any smooth function $f \in C^\infty(M)$, we can form the composition

$$f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$$

and take its derivative at $t = 0$:

$$(f \circ \gamma)'(0) \in \mathbb{R}.$$

This gives us a *linear functional* on $C^\infty(M)$, i.e. a linear map to the real numbers, which we denote by

$$\begin{aligned} w_\gamma : C^\infty(M) &\to \mathbb{R} \\ f &\mapsto (f \circ \gamma)'(0). \end{aligned}$$

This further satisfies the **Leibniz rule**

$$w_\gamma(f \cdot g) = w_\gamma(f) g(x) + f(x) w_\gamma(g). \tag{6.2}$$

Let's recall the proof. Since $\gamma(0) = x$, we have:

$$
\begin{aligned}
w_\gamma(f \cdot g) &= \lim_{t\to 0} \frac{f(\gamma(t))g(\gamma(t)) - f(x)g(x)}{t} \\
&= \lim_{t\to 0} \frac{f(\gamma(t))g(\gamma(t)) - f(x)g(\gamma(t)) + f(x)g(\gamma(t)) - f(x)g(x)}{t} \\
&= \lim_{t\to 0} \frac{\left(f(\gamma(t)) - f(x)\right) g(\gamma(t)) + f(x)\left(g(\gamma(t)) - g(x)\right)}{t} \\
&= w_\gamma(f)g(x) + f(x)w_\gamma(g).
\end{aligned} \tag{6.3}
$$

A linear functional on $C^\infty(M)$ that also satisfies (6.2) is called a **derivation at** $x$. For our second definition, we take

$$T_x^{(2)}M := \text{space of all derivations at } x.$$

As explained above, the path $\gamma$ defines an element $w_\gamma$ of this space which we can think of as $\gamma'(0)$. Also notice that since one can add and scalar multiply derivations, $T_x^{(2)}M$ is automatically a vector space.

**Example 6.2.** Fix coordinates $\{x^i\}$ as before. Define the derivation

$$\frac{\partial}{\partial x^i} : g \mapsto \frac{\partial\left(g \circ \varphi^{-1}\right)}{\partial x^i} = \lim_{t\to 0} \frac{g(x^1, \dots, x^i + t, \dots, x^n) - g(x^1, \dots, x^n)}{t}.$$

This is checked to be a derivation as in (6.3). In fact, letting $\gamma_i(t)$ be the path such that

$$\varphi\left(\gamma_i(t)\right) = (x^1, \dots, x^i + t, \dots, x^n)^T,$$

we have

$$\frac{\partial}{\partial x^i} = w_{\gamma_i}$$

by definition. Moreover, by the calculation in the last example, for any path $\gamma$, we have

$$w_\gamma = \sum_j \frac{dx^j(\gamma(t))}{dt} \frac{\partial}{\partial x^j}.$$

Hence the elements $\frac{\partial}{\partial x^i}$ span the image of $w_\gamma$ inside the space of derivations $T_x^{(2)}M$. Last, given another coordinate system $\{y^j\}$, we have

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

by the chain rule.

**Answer 3.** Notice that the derivative of a function $f$ along the path $\gamma$ at $t = 0$ only depends on the values of $f$ near $x$. This motivates the following definition, which leads to our third answer.

The **ring of germs** of smooth functions at $x$ is defined to be

$$C_x^\infty(M) = \{(U, f) \mid f \in C^\infty(U), U \ni x\} / \sim,$$

where $(U, f) \sim (V, y)$ iff there exists $x \in W \subset U \cap V$ such that

$$f|_W = g|_W.$$

The path $\gamma$ clearly gives a well-defined derivation $w_\gamma$ on germs, given by

$$w_\gamma\left([(U,f)]\right) = (f\circ\gamma)'(0) \in \mathbb{R}.$$

We will henceforth omit the set $U$ and brackets and simply denote a germ $[(U,f)]$ by $f$.

Now let

$$m_x = \{f \in C^\infty_x(M) \mid f(x) = 0\}.$$

This is an ideal in $C^\infty_x(M)$—in fact, as you will show on homework, it is the unique maximal ideal. So $C^\infty_x(M)$ is a *local ring*...hence the term!

The derivation $w_\gamma$ on $C^\infty_x(M)$ restricts (uniquely) to a linear functional on $m_x$. Since it is a derivation, if $f(x) = 0 = g(x)$ then $w_\gamma(f\cdot g) = 0$. Hence $w_\gamma$ restricts to zero on the square ideal $m_x^2$. The image of $w_\gamma$ therefore makes sense in

$$\left(m_x/m_x^2\right)^* =: T_x^{(3)}M.$$

**Theorem 6.3.** *The tangent spaces $T_x^{(i)}M$ are all canonically isomorphic. In particular, the tangent space at $x$, denoted henceforth by $T_xM$, is a real vector space of dimension $n$.*

You will prove this on your homework. The only tricky part is to show that there are no "extra" derivations beyond those spanned by $\frac{\partial}{\partial x^i}$ in coordinates. You may make use of the following result, which establishes the case of a ball in $\mathbb{R}^n$.

**Lemma 6.4.** *Let $w$ be a derivation at the origin on $B_r(0) \subset \mathbb{R}^n$.*[2] *If $f \in C^\infty(B_r(0))$ is such that $\frac{\partial f}{\partial x^i}\big|_0 = 0$ for $i = 1,\dots,n$, then $w(f) = 0$.*

*Proof.* This can be seen using the following version of Taylor's theorem.

For a twice-differentiable function $g(t)$ on $[0,1] \subset \mathbb{R}$, we have

$$g(1) - g(0) = \int_0^1 g'(t)dt.$$

We integrate by parts using the antiderivative $(t-1)$ of 1. This gives

$$\begin{aligned} g(1) - g(0) &= (t-1)g(t)\big|_0^1 - \int_0^1 (t-1)g''(t)dt \\ &= g'(0) + \int_0^1 (1-t)g''(t)dt. \end{aligned}$$

Now, given a $C^2$ function $f$ on a neighborhood of $0 \in \mathbb{R}^n$, and $x$ such that the line from 0 to $x$ is in the domain, we can define

$$g(t) = f(t\cdot x)$$

where $t$ scalar-multiplies the vector $x$. By the chain rule, we have

$$g'(t) = \sum_i x^i \frac{\partial f}{\partial x^i}(t\cdot x)$$

and

$$g''(t) = \sum_{i,j} x^i x^j \frac{\partial f}{\partial x^i \partial x^j}(t\cdot x).$$

[2]This was stated in class as a statement about derivations on the ring of germs, but the proof works for the ring of smooth functions on any domain which is star-shaped about the origin.

Substituting into the above formula, we obtain

$$f(x) = f(0) + \sum_i x^i \frac{\partial f}{\partial x^i} + \sum_{i,j} x^i x^j \int_0^1 (1-t) \frac{\partial f}{\partial x^i \partial x^j}(t \cdot x)\, dt. \tag{6.4}$$

We can use this formula to prove the Lemma. First of all, note that any derivation $w$ must vanish on constants, because

$$w(1) = w(1^2) = w(1) \cdot 1 + 1 \cdot w(1) = 2w(1).$$

Rearranging, we have $w(1) = 0$, whence $w(c) = c \cdot w(1) = 0$ for any constant $c$ as claimed. So $w$ vanishes on the first term on the RHS of (6.4). It vanishes on the second term by assumption. For any $f, g \in m_x$ and $h \in C^\infty(B_r(0))$, we must have $w(fgh) = 0$ by the Leibniz rule. Since the third term is a sum of smooth functions of this form, $w$ also vanishes on it. Hence $w(f) = 0$ as claimed. □

## 7. Derivative of a function, cotangent space, derivative of a map (Mon 9/23)

Last time we gave three equivalent definitions of the tangent space $T_x M$, motivated by the desire to define the derivative of $\gamma'(0)$ of a path through $x$. There is also a fourth definition, which is even more natural than the others.

**Answer 4.** Let

$$T_x^{(4)} M$$

denote the set of equivalence classes of smooth paths through $x$ under the relation:

$$\gamma_1 \sim \gamma_2 \Leftrightarrow (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0) \;\; \forall \;\; f \in C^\infty(M).$$

You will also show on homework that $T_x^{(4)} M$ is in canonical bijection with the other three. With this definition, we can simply take

$$\gamma'(0) = [\gamma] \in T_x^{(4)} M.$$

The only reason for not including $T_x^{(4)} M$ last time is that it does not obviously have the structure of a vector space, while the others do.

**Question.** Let $f$ be a smooth function on a neighborhood of $x$. What is the derivative of $f$ at $x$?

**Answer.** Given $v \in T_x M$ we have $v = \gamma'(0)$ for some path $\gamma$ through $x$. Then the function $f$ defines a map

$$v \mapsto (f \circ \gamma)'(0) \in \mathbb{R},$$

which is independent of the path $\gamma$ for which $\gamma'(0) = v$ (in $T_x^{(4)} M$, this is just by definition!). We call this map

$$df \in (T_x M)^* =: T_x^* M,$$

where the dual space $T_x^* M$ is called the **cotangent space** of $x$ at $M$. Since this is the dual of $T_x M$, it also has dimension $n$.

**Example 7.1.** Let $x^i$ be a coordinate system. By definition, the differential of $x^i$ is given by

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i{}_j.$$

Therefore $\{dx^i\}_{i=1}^n$ is the *dual basis* of $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$.

Given a smooth function $f$, we have

$$df\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial f}{\partial x^i},$$

which implies

$$df = \Sigma_j \frac{\partial f}{\partial x^j} dx^j. \tag{7.1}$$

In particular, for another coordinate system $y^j$, we have

$$dy^i = \Sigma_j \frac{\partial y^i}{\partial x^j} dx^j,$$

so

$$df = \Sigma_i \frac{\partial f}{\partial y^i} dy^i = \Sigma_{i,j} \frac{\partial f}{\partial y^i}\frac{\partial y^i}{\partial x^j} dx^j. \tag{7.2}$$

Since the coefficients of $dx^j$ in (7.1) and (7.2) must be equal, we have

$$\frac{\partial f}{\partial x^j} = \sum_i \frac{\partial f}{\partial y^i}\frac{\partial y^i}{\partial x^j}, \tag{7.3}$$

which agrees with the chain rule.

**Question.** What is the derivative of a smooth map $F : M \to N$?

In the case of $M = \mathbb{R}^m, N = \mathbb{R}^n$, recall the definition

$$dF_p = L <=> \lim_{x \to p} \frac{F(x) - F(p) - L(x-p)}{|x-p|} = 0.$$

By considering this limit along the $x^j$-axes, one proves that $L$ must be the Jacobian matrix of partial derivatives: $\left(\frac{\partial y^i}{\partial_{x_j}}\right)$. Of course, we could define the derivative of a general smooth map $F$ by pre/post-composing with coordinate charts and showing that the resulting map is well-defined on equivalence classes, per definition $T_x^{(1)}M$. But that would not be very satisfying. It is easiest to define $dF_p$ by reference to the fourth definition:

$$\begin{aligned} dF_x : T_x^{(4)}M &\to T_x^{(4)}N \\ [\gamma] &\mapsto [F \circ \gamma]. \end{aligned} \tag{7.4}$$

Since $F$ is defined locally by the smooth component functions $y^i \circ F(x)$, it is obvious that this map is well-defined.

The derivative of $F$ is even-more-obviously-well-defined at the level of derivations:

$$\begin{aligned} dF_x : T_x^{(2)}M &\to T_x^{(2)}N \\ w &\mapsto (g \mapsto w(g \circ F)). \end{aligned}$$

Here one just has to check that the pushforward of a derivation at $x$ is a derivation at $F(y)$, i.e. the Leibniz rule still holds. That can be done in two lines (as we did in class).

The derivative $dF_x$ is sometimes denoted by $F_*$ and called the "pushforward." This is for symmetry with the "pullback" map on cotangent vectors:

$$\begin{aligned} F^* : T^*_{F(x)}N &\to T^*_x M \\ \alpha &\mapsto (v \mapsto \alpha(dF_x(v)))\,. \end{aligned}$$

Note that $F^*$ is simply the adjoint of $dF_x = F_*$.

**Example 7.2.** Let $\{x^j\}$ be local coordinates on $M$ near $x$ and $\{y^j\}$ be local coordinates on $N$ near $f(x)$. After pre/post-composing with the charts, we can write

$$F(x) = \begin{pmatrix} y^1(x^1, \dots, x^m) \\ \cdots \\ y^n(x^1, \dots, x^m) \end{pmatrix}.$$

Then

$$dF_x\left(\frac{\partial}{\partial x^j}\right)(y^i) = \frac{\partial y^i}{\partial x^j}$$

and

$$dF_x\left(\frac{\partial}{\partial x^j}\right)(f) = \frac{\partial f}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j}\frac{\partial f}{\partial y^i}.$$

We can therefore write

$$dF_x\left(\frac{\partial}{\partial x^j}\right) = \sum_i \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i}$$

and

$$F^* dy^i = \sum_j \frac{\partial y^i}{\partial x^j} dx^j.$$

The main goal of **Part 3** will be to show that if $dF_x$ is injective, surjective, or bijective, then $F$ is injective, surjective, or bijective, respectively, on a neighborhood of $x$. Maps with these properties are respectively called immersions, submersions, and local diffeomorphisms.

## 8. The tangent bundle and vector fields (Mon 9/23)

**Question.** How to make sense of a "smoothly varying family of tangent vectors?" I.e. a collection

$$X = \{X_x \in T_x M\}_{x \in U}$$

for which $X_x$ "varies smoothly."

**Answer 1.** Such a collection can be considered smooth if

$$X_x = \sum_i X^i(x) \frac{\partial}{\partial x^i}$$

for smooth functions $X^i(x)$ in local coordinates.

**Answer 2.** Define the **tangent bundle**

$$TM = \bigsqcup_{x \in M} T_x M \tag{8.1}$$

as a set. We'll employ the smooth manifold chart lemma to give smooth charts canonically attached to smooth charts on $M$, in such a way that the projection map

$$\pi : TM \to M$$

is smooth.

**Proposition 8.1.** *For any smooth n-manifold M; the tangent bundle $TM$ has a natural topology and smooth structure that make it into a 2n-dimensional smooth manifold. With respect to this structure, the projection $\pi : TM \to M$ is smooth.*

*Proof.* We begin by defining the maps that will become our smooth charts. Given any smooth chart $(U, \varphi)$ for $M$, note that $\pi^{-1}(U) \subset TM$ is the set of all tangent vectors to $M$ at all points of $U$. Let $(x^1(p), ..., x^n(p))$ denote the coordinate functions of $\varphi$, and define a map $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\tilde{\varphi}\left(\sum_i v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n).$$

Its image set is $\varphi(U) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. It is a bijection onto its image, because its inverse can be written:

$$\tilde{\varphi}^{-1}((x^1(p), ..., x^n(p), v^1, ..., v^n) = \sum_i v^i \frac{\partial}{\partial x^i}|_{\phi^{-1}}.$$

Now suppose we are given two smooth charts $(U, \phi)$ and $(V, \psi)$ for $M$, and let $(\pi^{-1}(U), \tilde{\varphi}$, $(\pi^{-1}(V), \tilde{\psi}$ be the corresponding charts on $TM$. The sets:

$$\tilde{\varphi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \varphi(U \cap V) \times \mathbb{R}^n$$

and

$$\tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \psi(U \cap V) \times \mathbb{R}^n$$

are open in $\mathbb{R}^n$, and the transition map $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ can be written explicitly as:

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(x^1, ..., x^n, v^1, ..., v^n) = \left(\psi(\varphi^{-1}(x^1, \ldots, x^n)), d_{\varphi^{-1}(x)}\left(\psi \circ \varphi^{-1}\right)(v^1, \ldots, v^n)\right), \tag{8.2}$$

which is smooth.

Choosing a countable cover $\{U_i\}$ of $M$ by smooth coordinate domains, we obtain a countable cover of $TM$ by coordinate domains $\{\pi^{-1}(U)\}$ satisfying conditions (1) – (4) of the smooth manifold chart lemma. To check the Hausdorff condition, just note that any two points in the same fiber of $\pi$ lie in one chart, while if $(p, v)$ and $(q, w)$ lie in different fibers, there exist disjoint, smooth coordinate domains $U$, $V$ for $M$ such that $p \in U$ and $q \in V$, and then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint coordinate neighborhoods containing $(p, v)$ and $(q, w)$, respectively. To see that $\pi$ is smooth, note that with respect to the charts $(U, \phi)$ for $M$ and $(\pi^{-1}, \tilde{\phi})$ for $TM$, its coordinate representation is $\pi(x, v) = x$ □

**Definition 8.2.** A smooth vector field over $U$, $X \in \mathfrak{X}(U)$, is a smooth section of

$$\pi : U \mapsto TM,$$

i.e., a smooth map such that

$$\pi \circ X = \mathrm{Id}_U,$$

or in other words $X(x) \in T_x M$ for all $x \in U$.

Note that by letting $X$ act on functions at each point $x \in U$, $X \in \mathfrak{X}(U)$ defines a **derivation**:

$$X : C^\infty(U) \to C^\infty(U)$$
$$f \mapsto X(f) = df(X),$$

i.e. a linear map satisfying the Leibniz rule

$$X(fg) = X(f)g + fX(g).$$

**Question.** What is the derivative of a vector field?

This question will be partially addressed in **Part 4** and more fully addressed in Math 765.

# Part 3. Immersions, submersions, submanifolds

## 9. The inverse function theorem (Wed 9/25)

As mentioned last week, the goal is to go from information on the derivative $dF$ at a point (injectivity, surjectivity, bijectivity) to the same information on $F$ in a neighborhood. We start with bijectivity, as it turns out that the other cases can be reduced to this one.

**Theorem 9.1** (Inverse function theorem)**.** *Let $F : M \to N$ be a smooth map and $p \in M$. Suppose $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism. Then there exist neighborhoods $U_0 \ni p$ and $V_0 \ni F(p)$ such that the restriction*

$$F|_{U_0} : U_0 \to V_0$$

*is a diffeomorphism.*

*Proof.* Since the statement is local, without loss of generality, we can replace $M$ by a coordinate neighborhood $U$ centered at $p$ and $N$ by a coordinate neighborhood $V$ centered at $F(p) \in N$. The statement is clearly true if and only if it is true after pre/post composing with the coordinate maps. So we may assume without loss of generality that $U$ and $V$ are subsets of $\mathbb{R}^n$, with $p = 0 = F(p)$. (Since $dF_p$ is an isomorphism, the tangent spaces are of the same dimension, so $M$ and $N$ are of the same dimension $n$.)

Let

$$L = (dF)_0.$$

Given $y \in V$ sufficiently close to zero, we shall use **Newton's method** to find $x = F^{-1}(y)$. This means that we "adjust" a candidate solution, $x$, by following the linear approximation until it hits the value $y$. The adjusted $x$, which we call $T_y(x)$, is determined by

$$L\left(T_y(x) - x\right) = y - F(x)$$

and

$$T_y(x) = x + L^{-1}\left(y - F(x)\right).$$

Here we have used the assumption that $L$ is an isomorphism in order to apply the inverse $L^{-1}$ to both sides.

Notice that clearly we have

$$y = F(x) \Leftrightarrow T_y(x) = x.$$

The problem of solving $F(x) = y$ is therefore reduced to finding a fixed point of the map $T_y$.

**Lemma 9.2** (Contraction mapping principle)**.** *Suppose that $X$ is a nonempty complete metric space and $T : X \to X$ satisfies*

$$d(T(x), T(y)) \leq \lambda d(x, y)$$

*for all $x, y \in X$ and some constant $\lambda < 1$. Then there exists a unique fixed point of $T$ on $X$.*

*Proof.* Exercise (you should have seen this before). □

We will apply the contraction mapping principle with $X = \overline{B_\delta(0)}$, the closure of a small ball around the origin, where $\delta$ is chosen as follows. We have

$$F(x) = Lx + G(x)$$

for a "remainder" function $G(x)$ which is again smooth with $dG_0 = 0$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|dG_x| < \varepsilon \tag{9.1}$$

for all $|x| \leq \delta$. Here we are taking the operator norm of a linear map:

$$|A| := \sup_{|v|=1} |Av|. \tag{9.2}$$

We now let $\delta > 0$ be such that (9.1) is satisfied with

$$\varepsilon = \frac{1}{2|L^{-1}|}.$$

By the mean-value theorem, (9.1) implies

$$|G(x') - G(x)| \leq \frac{1}{2|L^{-1}|}|x - x'| \tag{9.3}$$

for all $|x| \leq \delta$, as well as simply

$$G(x) \leq \frac{1}{2|L^{-1}|}|x|. \tag{9.4}$$

Assuming also that

$$|y| < \frac{\delta}{2|L^{-1}|}, \tag{9.5}$$

we have

$$\begin{aligned} |T_y(x)| &= |x + L^{-1}\left(y - Lx - G(x)\right)| \\ &= |L^{-1}(y - G(x))| \\ &\leq |L^{-1}|\left(|y| + \varepsilon|x|\right) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta, \end{aligned}$$

where we have applied (9.4-9.5). This shows that $T_y$ restricts to a map

$$T_y : \overline{B_\delta(0)} \to B_\delta(0) \subset \overline{B_\delta(0)}.$$

We claim that it is a contraction on $\overline{B_\delta(0)}$. Indeed, from (9.3), we have

$$\begin{aligned} |T_y(x) - T_y(x')| &= |x - x' + L^{-1}\left(y - Lx - G(x) - y + Lx + G(x)\right)| \\ &= |L^{-1}\left(G(x) - G(x')\right)| \\ &\leq |L^{-1}||G(x) - G(x')| \\ &\leq \frac{1}{2}|x - x'|. \end{aligned}$$

We conclude from the Lemma that given $y \in B_{\frac{\delta}{2|L^{-1}|}} =: V_0$, there exists a unique $x \in B_\delta(0)$ such that $F(x) = y$. (*A priori*, $x$ only lies in the closure $\overline{B_\delta(0)}$, but since it solves $T_y(x) = x$ and the image of $T_y$ is contained in $B_\delta(0)$, the same is true of $x$.) Letting

$$U_0 := F^{-1}(V_0) \cap B_\delta(0),$$

we have shown that $F : U_0 \to V_0$ has an inverse map $F^{-1}$.

It remains to check continuity and smoothness of $F^{-1}$. Write $y = F(x)$ and $y' = F(x')$. For continuity, we compute

$$\begin{aligned} |y' - y| &= |Lx' + G(x') - Lx - G(x)| \\ &\geq |L(x - x')| - |G(x') - G(x)| \\ &\geq \frac{1}{|L^{-1}|}|x' - x| - \frac{1}{2|L^{-1}|}|x' - x| \\ &= \frac{1}{2|L^{-1}|}|x' - x|. \end{aligned} \tag{9.6}$$

Here we have used (9.3) and the fact that $|L(x-x')| \geq \frac{1}{|L^{-1}|}|x - x'|$ (exercise). This expression certainly implies that $x' \to x$ as $y' \to y$, so $F^{-1}$ is continuous.

Next, write $L_x = dF_x$. After shrinking $\delta$, we can assume that $L_x$ is invertible for all $x \in U_0$. We claim that for $x \in U_0$, and $y = F(x)$, we have

$$\left(dF^{-1}\right)_y = L_x^{-1}. \tag{9.7}$$

We have

$$\begin{aligned} \lim_{y' \to y} \frac{F^{-1}(y') - F^{-1}(y) - L_x^{-1}(y' - y)}{|y' - y|} &= \lim_{y' \to y} L_x^{-1} \frac{L_x(x' - x) - (y' - y)}{|y' - y|} \\ &= \lim_{x' \to x} -\frac{|x' - x|}{|y' - y|} L_x^{-1} \frac{F(x') - F(x) - L_x(x' - x)}{|x' - x|}. \end{aligned}$$

It follows from (9.6) that $\frac{|x'-x|}{|y'-y|}$ is bounded, and the rest of the expression tends to zero by definition of $L_x$. Therefore the limit is zero, which proves (9.8).

To see that the first derivative is continuous, we rewrite (9.8) as

$$\left(dF^{-1}\right)_y = \left(L_{F^{-1}(y)}\right)^{-1}. \tag{9.8}$$

The function $(L_x)^{-1}$ is smooth in $x$ near zero, since it is the inverse of a smooth matrix-valued function. Since $F^{-1}(y)$ is $C^0$, this implies that $\left(L_{F^{-1}(y)}\right)^{-1}$ is continuous in $y$. But this implies that the LHS of (9.8) is continuous in $y$; in other words, $F^{-1}(y)$ is $C^1$.

Now, the fact that $F^{-1}(y)$ is $C^1$ implies that $\left(L_{F^{-1}(y)}\right)^{-1}$ is also $C^1$, since a composition of a smooth and a $C^1$ function is $C^1$. But then the LHS of (9.8) is $C^1$, which is to say, $F^{-1}(y)$ is $C^2$. Continuing in this way, we obtain continuity of all higher derivatives of $F^{-1}(y)$. □

## 10. Local diffeomorphisms and covering maps (Wed 9/25)

**Definition/Lemma 10.1.** *A smooth map $F : M \to N$ is called a* **local diffeomorphism** *if $dF_x$ is an isomorphism for all $x \in M$. In particular, for $x \in M$, there exists $U_0 \ni x$ and $V_0 \ni F(x)$ such that*

$$F|_{U_0} : U_0 \to V_0$$

*is a diffeomorphism.*

This definition has a close relationship with the following topological one.

**Definition 10.2.** A map $\pi : X \to Y$ between two topological spaces is called a (topological) **covering map** is every point $q \in Y$ has a neighborhood $V$ that is *evenly covered*, i.e.

$$\pi^{-1}(V) = \sqcup \tilde{V}_\alpha$$

for some disjoint open sets $\tilde{V}_\alpha \subset X$ such that $\pi|_{\tilde{V}_\alpha} : \tilde{V}_\alpha \to V$ is a homeomorphism for each $\alpha$.

**Example 10.3.** Let $k \in \mathbb{N} \cup \{\infty\}$. Consider the projection map

$$\pi : \mathbb{R}/2\pi k\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z} = S^1$$
$$[\theta] \mapsto [\theta].$$

Not only is this a local diffeomorphism (for tautological reasons), it is a $k$-to-1 covering map.

**Example 10.4.** Consider the projection map $\pi : S^n \to \mathbb{RP}^n = S^n/\pm 1$. This is a local diffeomorphism, since the hemisphere charts $U_i^\pm$ are each mapped diffeomorphically onto the standard coordinate chart $U_i$. This also shows immediately that $\pi$ is a 2-to-1 covering map.

**Example 10.5.** To make a stupid example of a local diffeomorphism that is *not* a covering map, take $M = (0,1) \sqcup (-1,1)$, $N = (-1,1)$, and $\pi$ the obvious map. Then 0 is not evenly covered.

Here are some simple theorems about local diffeomorphisms versus covering maps that we did not get to discuss in class, but which may come up again later. The first gives a criterion for when a local diffeomorphism is a covering map.

**Proposition 10.6** (Lee, Prop 4.46)**.** *Let $F : M \to N$ be a local diffeomorphism which is* **proper** *(i.e. the inverse image of compact subset is compact), and assume $N$ is connected. Then $N$ is a covering map.*

*Proof.* The main point is to observe that the fiber over any point is a discrete set, by the local diffeomorphism property; but this set must be compact by properness, and a discrete set is compact if and only if it is finite. So, to obtain a neighborhood $V$ which is evenly covered, one need only take the intersection of finitely many open sets on which $F$ is a local diffeomorphism. □

The second one is about when a topological covering map can be promoted to a local diffeomorphism.

**Proposition 10.7** (Lee, Prop. 4.40)**.** *Suppose that $\pi : X \to N$ is a covering map, with $N$ a smooth manifold. There exists a unique smooth structure on $X$ such that $\pi$ is a local diffeomorphism.*

*Proof.* One can endow $X$ with an atlas consisting of connected components of the preimages of evenly covered coordinate balls in $N$. These will overlap only if their images in $N$ overlap, in which case the transition functions are exactly the same. □

One consequence is that the universal cover of a smooth manifold is naturally a smooth manifold.

There is another one on your homework showing that given a covering action by diffeomorphisms, you can "descend" the smooth structure to the quotient in a unique way:

**Proposition 10.8.** *Suppose that a group $G$ acts properly discontinuously*[3] *by diffeomorphisms on a smooth manifold $M$. Then $M/G$ has a unique smooth structure such that the projection $\pi : M \to M/G$ is a local diffeomorphism.*

*Proof.* Homework 4. □

For example, this gives you another way to define the standard smooth structure on $\mathbb{RP}^n$.

## 11. Immersions and embeddings (Fri 9/27)

11.1. **Immersions.** A smooth map $F : M \to N$ is an **immersion** if $dF_x$ is injective for all $x \in M$. (Note that necessarily $m = \dim(M) \geq n = \dim(N)$.)

**Remark 11.1.** $dF$ being injective is an *open condition*, meaning that if $dF_p$ is injective then there exists an open neighborhood $U \ni p$ such that $dF_x$ is injective for all $x \in U$. There are many ways to see this. For instance, by Gaussian elimination, the matrix representing $dF_p$ is injective if and only if the determinant of an $m \times m$ minor does not vanish, which is an open condition since the determinant is a polynomial in the coefficients. More geometrically, one can observe that since the $n$-sphere is compact, $dF_p$ is injective if and only if

$$\inf_{|v|=1} |dF_p(v)| > 0,$$

and by uniform continuity, this remains true after perturbing $dF_p$ slightly (exercise).

In the proof of the inverse function theorem (before 9.8) we used the case $m = n$ of this fact, which is easier to see since it's just nonvanishing of the determinant.

This observation also follows from the next result.

**Proposition 11.2.** *Suppose $dF_p$ is injective. There exists a coordinate system on $N$ near $F(p)$ such that $F$ takes the form*

$$F(x^1, \ldots, x^m) = (x^1, \ldots, x^m, \overbrace{0, \ldots, 0}^{n-m}).$$

*Proof.* We can suppose without loss of generality that the first $m \times m$ minor of $dF_p$ is nonsingular:

$$dF_p = \begin{pmatrix} \overbrace{*}^{\text{nonsingular}} \\ * \end{pmatrix}.$$

For, we know that $m$ of the rows must be linearly independent, so we can simply permute the coordinates on $N$ so that these rows become the first rows of $dF_p$.

Now, by shrinking $U$, we may suppose that $F(U) \Subset V$, and let $\varepsilon > 0$ be such that $d(F(U), V^c) > \varepsilon$. We can then define a map

$$\begin{aligned} \bar{F} : U \times B^{n-m}_{\varepsilon}(0) &\to V \subset \mathbb{R}^n \\ (x, y) &\mapsto F(x) + (0, y). \end{aligned}$$

[3] An action $G \circlearrowright M$ is called *properly discontinuous* if each $p \in M$ has a neighborhood $U \ni p$ such that $U \cap g(U) = \varnothing$ for $g \neq e \in G$.

Here we are using the addition operation on $\mathbb{R}^n$; written out fully, the above expression means

$$\bar{F}(x,y) = \begin{pmatrix} F^1(x^1,\dots,x^m) \\ \vdots \\ F^m(x^1,\dots x^m) \\ F^{m+1}(x^1,\dots,x^m) + y^1 \\ \vdots \\ F^n(x^1,\dots,x^m) + y^{n-m} \end{pmatrix}.$$

The derivative of $\bar{F}$ is a matrix of the form

$$d\bar{F}_p = \begin{pmatrix} * & 0 \\ * & \mathbf{1} \end{pmatrix},$$

where the left part is unchanged. From the form of $d\bar{F}_p$, it is easy to show that the columns of $d\bar{F}_p$ are linearly independent. Since this is an $n \times n$ matrix, it is invertible. By the inverse function theorem, there exists an inverse

$$\bar{F}^{-1} : V_0 \to U_0 \subset U \times B_\varepsilon(0) \subset \mathbb{R}^n$$

which is also a diffeomorphism. As our new coordinate chart, defined on $V_0 \subset N$, we take

$$\psi := \bar{F}^{-1} : V_0 \to U_0 =: \hat{V}_0.$$

To see that this does the trick note that since $\bar{F}^{-1} \circ \bar{F} = \mathrm{Id}$, we have

$$(x,y) = \psi(\bar{F}(x,y)) \quad \forall (x,y) \in V_0.$$

Taking $y = 0$, we have

$$(x,0) = \psi(\bar{F}(x,0))$$

But from the definition of $\bar{F}$, we have $\bar{F}(x,0) = F(x)$. So we obtain

$$(x,0) = (\psi \circ F)(x),$$

which is just the desired expression. □

11.2. **Embeddings.** An immersion $F : M \to N$ is called an **embedding** if it is a homeomorphism onto its image.

**Remark 11.3.** The following is a necessary and sufficient condition for a continuous map from a T1 space to be a homeomorphism onto its image: for every $p \in M$ and neighborhood $U \ni p$, there exists a neighborhood $V \ni F(p)$ such that the inverse image $F^{-1}(V) \subset U$.

**Example 11.4** (Immersion, not injective)**.**

$$t \mapsto \left(\frac{1}{2}t + \cos t, \sin t\right).$$

The image is a doodle crossing itself once per cycle.

**Example 11.5** (Homeomorphism onto its image, not an immersion)**.**

$$t \mapsto (t + \cos t, \sin t).$$

This is one-to-one but has a “cusp” each cycle.

**Example 11.6** (Injective immersion, not homeomorphism onto its image)**.** The "figure eight" in $\mathbb{R}^2$ can be parametrized by $\mathbb{R}$ in such a way that $F(0) = (0,0)$ but also the limit as $x \to \pm\infty$ is zero. Letting $U$ be any bounded interval containing $0 \in \mathbb{R}$, the inverse image of any open set $V \ni (0,0)$ contains points near $\pm\infty$, so is not contained in $U$. By Remark 11.3, this is not an embedding.

**Remark 11.7.** Supposing that a map is an injective immersion, Lee Proposition 4.22 gives several handy criteria for the map to be an embedding: for instance, if $M$ is compact then this has to be the case, because a continuous map from a compact space to a Hausdorff space must be closed, so its inverse satisfies the "closed set" definition of continuity.

**Example 11.8** (Embedding)**.** Here is a standard way to embed the torus $T^2$ into $\mathbb{R}^3$ :

$$(\theta, \phi) \mapsto ((2 + \cos\phi)\cos\theta, (2 + \sin\phi)\sin\theta, \sin\phi)\,. \tag{11.1}$$

We have

$$\frac{\partial F}{\partial \theta} = \begin{pmatrix} -(2+\cos\phi)\sin\theta \\ (2+\cos\phi)\cos\theta \\ 0 \end{pmatrix}$$

and

$$\frac{\partial F}{\partial \phi} = \begin{pmatrix} -\sin\phi\cos\theta \\ -\sin\phi\sin\theta \\ \cos\phi \end{pmatrix}.$$

So the Jacobian matrix is

$$\begin{pmatrix} -(2+\cos\phi)\sin\theta & -\sin\phi\cos\theta \\ (2+\cos\phi)\cos\theta & -\sin\phi\sin\theta \\ 0 & \cos\phi \end{pmatrix}.$$

The determinant of the top $2 \times 2$ minor is

$$(2+\cos\phi)\sin\phi\left(\sin^2\theta + \cos^2\theta\right) = (2+\cos\phi)\sin\phi.$$

This is nonzero as long as $\phi \neq 0, \pi + 2\pi\mathbb{Z}$, so we are done apart from those cases. For $\phi = 0, \pi + 2\pi\mathbb{Z}$, the Jacobian is

$$\begin{pmatrix} -(2\pm1)\sin\theta & 0 \\ (2\pm1)\cos\theta & 0 \\ 0 & \pm1 \end{pmatrix}.$$

The columns are linearly independent for any value of $\theta$, so we are done.

**Remark 11.9.** Note that in the last example we were slightly casual in that we treated $\theta$ and $\phi$ as coordinates on $T^2$ although they are not globally well-defined. Really, what we showed is that the map $\mathbb{R}^2 \to \mathbb{R}^3$ defined by the formula (11.1) is an immersion. This does of course descend to a map from the torus $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and the map is smooth because the projection $\mathbb{R}^2 \to T^2$ is a local diffeomorphism—for the general statement of this smooth descent property, see Corollary 12.8 in the next lecture.

## 12. Submersions and the constant rank theorem (Fri 9/27-Mon 9/30)

12.1. **Submersions.** A smooth map $F : M \to N$ $(m \geq n)$ is called a **submersion** if $dF_p$ is surjective for all $p \in M$.

**Remark 12.1.** Being surjective is also an open condition on a linear map. This can also be seen by considering minors / using Gaussian elimination. Alternatively, one can observe that $dF_p = F_* : T_pM \to T_{F(p)}N$ is injective if and only if $F^* : T^*_{F(p)}N \to T^*_pM$ is injective.

**Proposition 12.2.** *Suppose $dF_p$ is surjective. There exist coordinates on $M$ near $p$ such that $F$ takes the form*

$$F(x^1, \dots, x^n, y^1, \dots, y^{m-n}) = (x^1, \dots, x^n).$$

**Remark 12.3.** As we shall see from the proof, one only has to change $n$ of the variables on the domain and can leave $m - n$ of them unchanged (the ones that will be labeled as $y^1, \dots, y^{m-n}$). See the corollary for a statement to this effect.

*Proof.* Since the theorem is local, take open sets $p \in U \subset M$ and $F(p) \in V \subset N$. By permuting the coordinates on the domain, we may assume that

$$dF_p = \left( \overset{n}{*} \;\middle|\; \overset{m-n}{*} \right)$$

where the left $n \times n$ block is nonsingular. Now, consider the map

$$\begin{aligned} \bar{F} : U &\to V \times \mathbb{R}^{m-n} \\ (x, y) &\mapsto (F(x, y), y). \end{aligned}$$

We have

$$d\bar{F}_p = \begin{pmatrix} * & * \\ 0 & I_{m-n} \end{pmatrix}$$

where the left part is the same. As before, it is easy to see that the rows are linearly independent, so this is invertible. The inverse function theorem guarantees the existence of an inverse $\bar{F}^{-1}$ on a smaller domain, which we write in the form

$$\bar{F}^{-1}(x, y) = (A(x, y), B(x, y))$$

for smooth functions $A(x, y) \in \mathbb{R}^n$ and $B(x, y) \in \mathbb{R}^{m-n}$. For any point $(x, y)$, we have

$$\begin{aligned} (x, y) &= \bar{F}(\bar{F}^{-1}(x, y)) \\ &= \bar{F}(A(x, y), B(x, y)). \end{aligned}$$

Comparing both sides, we see that $B(x, y) = y$, so in fact we have

$$\begin{aligned} (x, y) &= \bar{F}(A(x, y), y) \\ &= (F(A(x, y)), y), y), \end{aligned}$$

by definition $\bar{F}$. We therefore have

$$x = F(A(x, y), y).$$

Take new coordinates on $U_0 \subset U \subset M$ given by $\varphi(x, y) : (A(x, y), y) \in \mathbb{R}^m$. We then have

$$(F \circ \varphi)(x, y) = F(A(x, y), y) = x,$$

as desired. □

As a corollary of the proof, we have:

**Corollary 12.4** (Classical Implicit Function Theorem). *Suppose we have a smooth function* $F : U \times V \subset \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$ *with* $F(0) = 0$ *and let* $(x, y) = (x^1, \cdots, x^n, y^1, \cdots, y^{m-n})$ *and* $z = F(x, y)$. *Suppose that the matrix of partials*

$$\left( \frac{\partial z^i}{\partial x^j}\Big|_0 \right)_{i,j=1}^n$$

*is nonsingular. Then for each* $z$ *sufficiently small, there exists a smooth function* $\phi_z(y)$ *such that*

$$F(\phi_z(y), y) = z.$$

*Proof.* This follows by letting $\varphi_z(y) = A(y, z)$ in the previous proof. □

**Remark 12.5.** It is good to try and draw a picture of this statement (as we did in class) and think about how to prove it directly using Newton's method.

### 12.2. **Examples of submersions.**

- Any local diffeomorphism is a submersion with $m = n$, and vice-versa. For instance any smooth covering map, e.g. our favorites $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = T^2$ and $S^n \to S^n/\pm 1 = \mathbb{RP}^n$.

- Let $K = \mathbb{R}$ or $\mathbb{C}$. The canonical projection

  $$\pi : K^{n+1} \setminus \{0\} \to K\mathbb{P}^n$$

  is a submersion (homework).

- Let $\pi_0$ be the restriction of $\pi$ to the unit sphere in $K^{n+1} \setminus \{0\}$. In the case $K = \mathbb{R}$, the restriction is a local diffeomorphism, as we know. In the case $K = \mathbb{C}$, you will prove on your homework that the differential is still surjective after restriction to $S^{2n+1} \subset \mathbb{C}^{n+1}$, so remains a submersion.

  Notice that the fiber over each point is a circle: given $X = (X^1, \ldots, X^{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$, the fiber $\pi_0^{-1}[X]$ over the line $[X] =\in \mathbb{CP}^n$ is:

  $$\pi^{-1}\left[X^1, \ldots, X^{n+1}\right] = \{(\lambda X^1, \ldots, \lambda X^{n+1}) \mid \lambda \in \mathbb{C}, |\lambda| = 1\}.$$

  So the fibers are all diffeomorphic. This turns out to be a general property of submersions from compact to connected manifolds.

  This example is called the *Hopf fibration* and is usually represented by a diagram:

  $$S^1 \to S^{2n+1} \xrightarrow{\pi_0} \mathbb{CP}^n.$$

  In the case $n = 1$, by Proposition 4.4, we have

  $$S^1 \to S^3 \to S^2.$$

- (Homeomorphism, not submersion) The map
$$\begin{aligned} \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$
is surjective (indeed, a homeomorphism) but is not a submersion because the differential at zero is not surjective.

- (Not homeomorphism, not submersion) We can extend the previous example to the complex numbers:
$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^3. \end{aligned}$$
Notice that the fiber over any point $w \neq 0$ consists of three points, whereas the fiber over $w = 0$ consists of a single point. So over $\mathbb{C}$, the failure to be a submersion shows up not only in a drop in rank of the differential but in a change in the topology of the fibers.

12.3. **Properties of submersions.**

**Proposition 12.6.** *A submersion is an open map. A surjective submersion is a quotient map.*

*Proof.* The openness follows from Proposition 12.2. For, given a neighborhood $W \subset M$ and a point $y \in F(W)$, let $p$ such that $F(p) = y$. There exist neighborhoods $U \ni p$ and $V \ni y$ such that $F|_U$ takes the form of a projection onto a coordinate plane containing $V$; we may assume without loss of generality that $U \subset W$. In particular, $F(U) = V$, so $V \subset F(W)$. Since $y \in F(M)$ was arbitrary, we are done.

It is a fact that a continuous, open, surjective map is a quotient map. □

**Proposition 12.7.** *Let $M, N$, and $P$ be smooth manifolds. Suppose that we have a diagram of maps*
$$\begin{array}{ccc} M & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\tilde{F}} & \\ N & \xrightarrow{F} & P, \end{array}$$
*where $\pi$ is a (smooth) submersion and $F, \tilde{F}$ are continuous. Then $\tilde{F}$ is smooth if and only if $F$ is smooth.*

*Proof.* To check smoothness, it suffices to use a coordinate chart of the form guaranteed by Proposition 12.2. But then the claim is obvious, because smoothness of a map amounts to smoothness of all coordinate functions. □

**Corollary 12.8.** *Surjective submersions have the* **smooth descent property.** *I.e., given a smooth map $\tilde{F}$ as above which is constant on the fibers of $\pi$, then there exists a unique* **smooth** *map $F$ such that the diagram commutes.*

*Proof.* Since a surjective submersion is a quotient map, there exists a continuous map $F$ completing the diagram by the universal property of quotient maps. By the previous proposition, if $\tilde{F}$ is smooth, then so is $F$. □

**Remark 12.9.** This property is useful for doing your homework, especially in the case of local diffeomorphisms. For instance, to define a map from $T^n$ or $\mathbb{RP}^n$ into another space, you just have to make a smooth map from $\mathbb{R}^n$ or $S^n$, respectively, that is constant on fibers (i.e. equivalence classes).

Another application that may be helpful later on is:

**Theorem 12.10** (Uniqueness of smooth quotients)**.** *Suppose given a diagram of smooth manifolds and surjective submersions of the form:*

$$\begin{array}{ccc} & M & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ N_1 & \leftarrow - - - \quad - - - \rightarrow & N_2. \end{array}$$

*If $\pi_1$ is constant on the fibers of $\pi_2$ and vice-versa, then $N_1$ and $N_2$ are diffeomorphic by a map which completes the diagram.*

*Proof.* This follows by completing the diagram in both directions using Corollary 12.8. □

12.4. **The constant rank theorem.** We have the following generalization of the immersion and submersion theorems proven above.

**Theorem 12.11.** *Suppose that $dF_x$ has constant rank $k$ for each $x \in U \subset M$, and let $p \in U$. After changing coordinates near $p$ and $F(p)$, $F$ takes the local form*

$$F(x^1, \ldots, x^k, y^1, \ldots, y^{m-k}) = (x^1, \ldots, x^k, \overbrace{0, \ldots, 0}^{n-k}).$$

**Remark 12.12.** While being injective or surjective (i.e. having *maximal rank*) is an open condition, having constant rank less than the maximal one is *not* an open condition, because the rank can be lower at a point than it is at nearby points (due to a minor vanishing there). So the assumption of this theorem is stricter than the previous two.

*Proof.* Suppose that the top $k \times k$ block of $dF_p$ is nonsingular. Then the restriction to the coordinate plane $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^m$ is an immersion. By Proposition 11.2, we can choose coordinates on $N$ so that

$$F(x^1, \ldots, x^k, 0, \ldots, 0) = (x^1, \ldots, x^k, \overbrace{0, \ldots, 0}^{n-k}).$$

Meanwhile, the composition of $F$ with the projection onto $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ is a submersion. By Proposition 12.2, we can change the first $k$ coordinates on $M$ so that our map takes the form

$$F(x^1, \ldots, x^k, y^1, \ldots, y^{m-k}) = (x^1, \ldots, x^k, G(x, y)),$$

for some map $G(x,y)$ satisfying $G(x,0) \equiv 0$ (by our first choice).[4] We have

$$dF_x = \begin{pmatrix} \mathbf{1}^{k\times k} & 0 \\ \frac{\partial G^i}{\partial x^j} & \frac{\partial G^i}{\partial y^j} \end{pmatrix}.$$

But since the rank is exactly $k$ for all $x \in U$, the lower-right block must vanish identically. In particular, the function $G(x,y)$ is independent of $y$, so $G(x,y) = G(x,0) \equiv 0$ for all $(x,y)$. Hence, the map $F$ is already in the required form. □

## 13. Submanifolds (Wed 10/2)

13.1. **Submanifolds and embeddings.** Let $k \le m = \dim(M)$. A subset $S \subset M$ is a called a **submanifold**[5] of $M$ if for all $p \in S$ there exists a "slice chart" for $S$ through $p$, i.e., a coordinate chart $U$ for $M$ containing $p$, together with constants $c^1, \ldots, c^{m-k}$, such that

$$\varphi(U \cap S) = \{\left(x^1, \ldots, x^k, c^1, \ldots, c^{m-k}\right) \in \hat{U}\}.$$

**Example 13.1.** $S^1 \subset \mathbb{R}^2$ is a submanifold. To make slice charts we can use polar coordinates: $(r,\theta) \in (0,\infty) \times (-\pi,\pi)$ or $(0,2\pi)$ maps to

$$(x,y) = (r\cos\theta, r\sin\theta).$$

To check that these are charts, one can either write down the inverse maps or check that the map and its differential at each point are bijective. The plane $r = 1$ in either chart corresponds to the intersection with $S^1$.

One can use the radius function in a similar way to extend any chart on $S^n$ into a slice chart for $S^n \subset \mathbb{R}^{n+1}$.

**Definition 13.2.** The **codimension** of a $k$-dimensional submanifold $S \subset M$ is $m - k$. A submanifold of codimension zero is an open subset; a submanifold of codimension $m$ is a discrete set of points. A submanifold of codimension one is called a **hypersurface.**

To legitimize the above definition, we need to show that a submanifold is in fact a manifold.

**Proposition 13.3.** *Let $\mathscr{A}$ be an atlas for $M$ with the property that for each $p \in S$, there exists $(U,\varphi) \in \mathscr{A}$ which is a slice chart for $S$ through $p$. Then the atlas*

$$\mathscr{A}_S := \{(U \cap S, \varphi|_S) \mid (U,\varphi) \in \mathscr{A} \text{ is a slice chart for } S\}$$

*is a smooth atlas for $S$.*

*Proof.* The atlas $\mathscr{A}_S$ covers $S$ by assumption and gives homeomorphisms to open subset of $k$-dimensional planes in $\mathbb{R}^n$, which we can identify with open sets in $\mathbb{R}^k$. So this is a topological atlas. To check smoothness of the transition maps, simply observe that a map is smooth if and only if its coordinate functions are smooth, and this remains true after restricting to coordinate planes. □

[4] One can (and should) also check that in the proof of Proposition 12.2, if $F(x,0) = x$, then $A(x,0) = x$, so that our previous choice is not messed up.

[5] Lee's book talks about something he calls "immersed submanifolds." We will not do that. Our definition is equivalent to Lee's definition of an "embedded submanifold," as we will show.

**Definition/Lemma 13.4.** *Let $S \subset M$ be a submanifold. The* **induced smooth structure** *defined by $\mathscr{A}_S$ above depends only on the smooth structure of $M$.*

*Proof.* We have to show that if $\mathscr{A}$ and $\mathscr{A}'$ are two equivalent atlases on $M$, then $\mathscr{A}_S$ and $\mathscr{A}'_S$ are both equivalent. But then $\mathscr{A} \cup \mathscr{A}'$ is an atlas, so by the previous proposition, the restriction $(\mathscr{A} \cup \mathscr{A}')_S = \mathscr{A}_S \cup \mathscr{A}'_S$ is also an atlas. We have shown that $\mathscr{A}_S$ and $\mathscr{A}'_S$ are equivalent atlases, i.e. define the same smooth structure. □

**Proposition 13.5.** *(a) Let $F : M \to N$. Then the restricted map $F|_S : S \to N$ is smooth.*

*(b) Let $F : N \to M$ with $F(N) \subset S$. Then $F$ is smooth if and only if the induced map $F : N \to S$ is smooth*

*Proof.* Since smoothness can be checked in any chart, both statements can be checked in adapted charts, where they are obvious. □

**Proposition 13.6.** *Let $F : N \to M$ be an embedding. Then $S = F(N)$ is a submanifold of $M$ and $F : N \to S$ is a diffeomorphism. Conversely, for a submanifold $S$, the inclusion map $S \hookrightarrow M$ is an embedding.*

*Proof.* Let $S = F(N)$. The map $F$ induces a homeomorphism from $N$ to $S$ by assumption. Proposition 11.2 shows that there exists a slice chart through every point, so $S$ is a submanifold. In the same slice chart, the inverse map is simply given by the identity, which is smooth.

The second statement is only claiming that the differential is injective; this is again obvious in a slice chart. □

**Corollary 13.7.** *Every submanifold is the image of a unique embedding up to diffeomorphism. I.e., if $S \subset M$ is a submanifold and there exist two embeddings $F_1 : N_1 \to M$ and $F_2 : N_2 \to M$ with $F_i(N_i) = S$ for $i = 1, 2$, then $N_1$ is diffeomorphic to $N_2$.*

*Proof.* We are given a diagram of maps:

$$\begin{array}{ccccc} N_1 & & & & \\ & \searrow & & & \\ & & S & \subset & M. \\ & \nearrow & & & \\ N_2 & & & & \end{array}$$

We have just shown that these arrows are smooth, and are in fact diffeomorphisms. So we can fill in the vertical arrow by a unique diffeomorphism. □

**Remark 13.8.** This result can be compared with Theorem 12.10 from last time.

### 13.2. **The fibers of a submersion are submanifolds.**

**Definition 13.9.** A submanifold $S \subset M$ is **properly embedded** if it is also a closed subset. (This is true iff the inclusion embedding $\iota : S \to M$ is a proper map, hence the terminology.)

**Example 13.10.**

- If $S \subset M$ is itself compact, then since $M$ is Hausdorff, it is automatically properly embedded.
- On your homework you are tasked with writing down the embedding of the Möbius strip in $\mathbb{R}^3$ that was drawn on the first day of class and again today. This is *not* a proper embedding, as one sees by considering the (missing) boundary of the strip.

**Theorem 13.11.** *Let $F : M \to N$ be a submersion. For each $q \in N$, the fiber $F^{-1}(q)$ is a properly embedded submanifold of $M$. If $F$ is a proper map then $F^{-1}(q)$ is compact. The same is true for constant rank maps.*

*Proof.* The fibers are closed since $F$ is continuous. It remains to construct slice charts. Given $q \in N$ and $p \in F^{-1}(q)$, we may work in a charts centered at $p$ and $q$. By Proposition 12.2/Theorem 12.11, there exists a chart on $M$ near $p$ such that $F$ takes the form

$$F(x^1, \dots, x^n, y^1, \dots, y^{m-n}) = (x^1, \dots, x^n).$$

We have

$$F^{-1}(q) = F^{-1}(0, \dots, 0) = \{(0, \dots, 0, y^1, \dots, y^{m-n})\}.$$

so this is a slice chart, as desired. (Note that the same chart works as a slice chart for fibers over nearby points in $N$.) ☐

**Example 13.12.** Let

$$F : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}_+$$
$$x \mapsto |x|^2 = x \cdot x.$$

We have

$$dF_x(v) = 2x \cdot v.$$

For $x \neq 0$, this is surjective onto $T\mathbb{R} = \mathbb{R}$, so a submersion. The level set is the sphere of radius $\sqrt{c}$:

$$F^{-1}(c) = S^n_{\sqrt{c}}.$$

So, finally, we can show that the sphere is a smooth manifold with only one line of calculation.

13.3. **The tangent space to a submanifold.** Given a submanifold $S \subset M$ and $p \in S$, since the inclusion map $\iota : S \to M$ is an embedding, it induces an injective map $d\iota_p : T_pS \hookrightarrow T_pM$. As with any injective map, it makes sense to identify $T_pS$ with its image inside $T_pM$, as we shall do.

**Proposition 13.13.** *The tangent space $T_pS \subset T_pM$ is characterized by:*

1. *$v \in T_pS \iff v = \gamma'(0)$ for some smooth path $\gamma : (-\varepsilon, \varepsilon) \to M$ whose image is contained entirely within $S$.*
2. *$v \in T_p$ if and only if for any neighborhood $p \in U \subset M$ and function $f \in C^\infty(U)$ which vanishes identically on $S \cap U$, we have $df(v) = 0$.*

3. *Suppose that $S = F^{-1}(q)$ is a fiber of a submersion. Then $T_pS = \ker dF_p$. In particular, we have an exact sequence of vector spaces*

$$0 \to T_pS \xrightarrow{d\iota_p} T_pM \xrightarrow{dF_p} T_{F(p)}N \to 0.$$

*Proof.* These are each obvious in a slice chart. □

Here is another one for later (used by most of you without proof on HW 4):

**Proposition 13.14.** *A smooth vector field on $M$ such that $X_p \subset T_pS$ for all $p \in S$ restricts to a smooth vector field on $S$.*

*Proof.* That the restriction defines a tangent vector on $S$ follows by the last proposition. Smoothness can be checked in any slice chart, as usual. □

## 14. Jacobian criteria (Fri 10/4-Mon 10/7)

In this section we will go over the methods that we currently have to construct submanifolds and add a few in the process. We begin by reformulating the definition of submanifold in the way it is used in practice.

**Definition/Lemma 14.1.** *Let $M$ be a manifold of dimension $m$, $S \subset M$ a subset, and $U \subset M$ an open set. A collection $f^1, \dots, f^{m-n} \in C^\infty(U)$ are called (local)* **defining functions** *for $S$ on $U$ if*

- *$S \cap U = \{f^i(x) = c^i \ \forall\, i\}$, for some constants $c^1, \dots, c^{m-n}$*
- *For each $p \in S \cap U$, the $(m-n) \times n$ Jacobian matrix*

$$\left( \frac{\partial f^i}{\partial x^j}\bigg|_p \right)$$

  *has full rank $m-n$. Here $x^j$ are any coordinates at $p$.*

*Then $S$ is a submanifold of $M$ if and only if there exist defining functions for $S$ near each point $p \in S$. This is called the* **Jacobian criterion** *for a submanifold.*

*In the case that $S \subset U$, these are called* **global defining functions.**

*Proof.* One can complete the collection of defining functions at $p$ to a slice chart in which $y^i = f^i$ for $i = 1, \dots, m-n$. This can either be proven directly using the inverse function theorem as usual, or one can define a map $\Phi : U \to \mathbb{R}^{m-n}$ by

$$\Phi(x) = \left(f^1(x), \dots, f^{m-n}(x)\right).$$

The Jacobian criterion just says that $d\Phi_p$ is surjective, so the existence of the remaining coordinates follows from Proposition 12.2 (with $x$ and $y$ reversed). □

We will now list four methods to construct submanifolds.

**1)** Write down a map $F : N \to M$ and show that it is a homeomorphism with injective differential (i.e. an embedding). For instance, we did this in Example 11.1 to produce an embedding $T^2 \to \mathbb{R}^3$. You have an embedding $\mathbb{RP}^2 \to \mathbb{R}^4$ on your homework.

**2)** Write down a collection of local or global defining functions and check the Jacobian criterion.

**Example 14.2.** Let $M = \mathbb{R}^3$ and define

$$S = \begin{Bmatrix} x^3 + y^3 + z^3 = 1 \\ x + y + z = 0 \end{Bmatrix}.$$

The Jacobian matrix is

$$\begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \\ 1 & 1 & 1 \end{pmatrix}.$$

The determinant of the first $2 \times 2$ minor is $3x^2 - 3y^2$. This vanishes iff $x = \pm y$. The determinant of the first and last one is $3x^2 - 3z^2$. This vanishes iff $x = \pm z$. So the Jacobian criterion fails where $y = \pm x$ and $z = \pm x$. But then the second equation reads

$$x \pm x \pm x = 0,$$

whose only solution is $x = 0$. Then also $y = z = 0$. Since the point $(0, 0, 0)$ does not lie on $S$, the Jacobian criterion is satisfied and $S$ is a 1-dimensional submanifold of $\mathbb{R}^3$.

**3)** Here is yet another formulation of the Jacobian criterion.

**Definition/Lemma 14.3.** *Let $F : M \to N$ be a smooth map. A point $q \in N$ is called a* **regular value** *of $F$ if $dF_p$ is surjective for all $p \in F^{-1}(q)$. In this case, if $S = F^{-1}(q)$ is nonempty, then it is a properly embedded submanifold of $M$ of codimension $n = \dim(N)$.*

*Proof.* The closedness follows because $\{q\} \subset N$ is a closed subset and $F$ is continuous. The fact that $S$ is a submanifold was proven verbatim in Theorem 13.11 above. □

Notice that **2)** is just **3)** in the case that $N = \mathbb{R}^n$ and the defining functions are global.

**Example 14.4.** A submersion is precisely a map for which every $q \in F(M) \subset N$ (which is an open subset) is a regular value.

**Example 14.5** (The sphere, again)**.** The map in Example 13.12 is also smooth at the origin, so goes from $\mathbb{R}^{n+1} \to \mathbb{R}$. All values are regular except $c = 0$. The fibers over $\mathbb{R}_+$ are spheres, and the fibers over $\mathbb{R}_-$ are empty, but the fiber over $c = 0$ is a point, which fails to be a submanifold of codimension one (hypersurface).

**Example 14.6.** Let $M = \mathbb{R}^2$. For $\lambda \in \mathbb{R}$, consider the polynomial $f_\lambda(x, y) = y^2 - x(x-1)(x-\lambda)$, and let

$$S_\lambda = \{f_\lambda(x, y) = 0\} \subset \mathbb{R}^2.$$

You will check on homework that this is smooth for $\lambda \neq 0, 1$. In class we drew a picture of this curve for different values of $\lambda$. For $\lambda > 0$, it is the union of an $S^1$ component passing through $(0,0)$ and $(1,0)$ and a noncompact component passing through $(\lambda, 0)$. As $\lambda \to 1$, this degenerates into the curve $S_1$ defined by

$$y^2 = x(x-1)^2$$

which has a singular point at $(1,0)$. In fact, $S_1$ is the image of the immersion

$$t \mapsto (t^2, t^3 - t).$$

For $0 < \lambda < 1$ and $\lambda < 0$, $S_\lambda$ is the union of an $S^1$ component through $(0,0)$ and $(\lambda, 0)$ and a noncompact component through $(1,0)$. The subset $S_0$ defined by

$$y^2 = x^2(x-1)$$

is the union of a noncompact component through $(1,0)$ and a single point at $(0,0)$, so is not even the image of an immersion (over the real numbers).

**The Jacobian criterion also works over** $\mathbb{C}$. We can prove the case of a single holomorphic defining function now; the case of several functions requires a bit more discussion as done for instance in a class on complex manifolds.

**Definition/Lemma 14.7.** *Suppose $f : \mathbb{C}^n \to \mathbb{C}$ is continuous and holomorphic in each variable. Fix $c \in \mathbb{C}$ and let $S = f^{-1}(c)$. Suppose that for each $p \in S$, $\frac{\partial f}{\partial z^j} \neq 0$ for some $j \in \{1, \dots, n\}$. Then $S$ is a submanifold of $\mathbb{C}^n$ of real codimension two, called a* **complex hypersurface***.*

*Proof.* We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by writing the coordinate functions as $z^j = x^j + iy^j$, for $j = 1, \dots, n$. By assumption, for some $j$, we have

$$0 \neq \frac{\partial f}{\partial z^j} = \lim_{\substack{t \to 0 \\ t \in \mathbb{C}}} \frac{f(z^1, \dots, z^j + t, \dots, z^n) - f(z^1, \dots, z^n)}{t}.$$

We argue as in the proof of the Cauchy-Riemann equations. Since this limit exists for $t \in \mathbb{R}$, we can take the limit along the real axis to obtain

$$\frac{\partial f}{\partial z^j} = \frac{\partial f}{\partial x^j}.$$

Since the limit also exists for $t \in i\mathbb{R}$, we also have

$$\frac{\partial f}{\partial z^j} = -i \frac{\partial f}{\partial y^j},$$

so $\frac{\partial f}{\partial y^j} = i \frac{\partial f}{\partial x^j}$ and both are nonzero. Hence, viewed as vectors in $\mathbb{R}^2 = \mathbb{C}$ (where $i$ acts by a $\pi/2$ rotation), they are linearly independent. □

**Example 14.8.** In the previous example, we could have taken $M = \mathbb{C}^2$ and considered $f_\lambda$ as a polynomial over $\mathbb{C}$. The Jacobian criterion is still true for $\lambda \neq 0, 1$, so we obtain a codimension-two submanifold of $\mathbb{C}^2$, i.e., a surface. This is called an **elliptic curve.**

**4)** Continuing in the algebraic-geometry vein, I'll show you how easy it is to construct smooth hypersurfaces in $K\mathbb{P}^n$. Let

$$F(X^1, \dots, X^{n+1})$$

be a homogeneous polynomial of degree $d$ in $n+1$ variables, and put

$$S = \left\{ \left[ X^1, \ldots, X^{n+1} \right] \in K\mathbb{P}^n \mid F(X^1, \ldots, X^{n+1}) = 0 \right\} \subset K\mathbb{P}^n.$$

Notice that this is well-defined due to the homogeneity of $F$: if $F(X^1, \ldots, X^{n+1}) = 0$ then

$$F(\lambda X^1, \ldots, \lambda X^{n+1}) = \lambda^d F(X^1, \ldots, X^{n+1}) = 0.$$

We first need a lemma:

**Lemma 14.9** (Euler's formula)**.** *For a homogeneous polynomial of degree $d$, we have the identity*

$$\sum_{i=1}^{n+1} X^i \frac{\partial F}{\partial X^i}(X) = d \cdot F(X).$$

*Proof.* It suffices to check the formula on a monomial $(X^1)^{d_1} \cdots (X^{n+1})^{d_{n+1}}$. For each $i$, we have

$$X^i \frac{\partial}{\partial X^i} (X^1)^{d_1} \cdots \left(X^{n+1}\right)^{d_{n+1}} = d_i (X^1)^{d_1} \cdots \left(X^{n+1}\right)^{d_{n+1}}.$$

Since $\sum d_i = d$, we get the formula. ◻

**Proposition 14.10.** *Suppose that for each $p \in S$, there exists $i \in \{1, \ldots, n+1\}$ such that*

$$\left. \frac{\partial F}{\partial X^i} \right|_p \neq 0.$$

*Then $S$ is a compact hypersurface in $K\mathbb{P}^n$, i.e. a submanifold of codimension $1$ or $2$ if $K = \mathbb{R}$ or $\mathbb{C}$, respectively.*

*Proof.* Let $U_i$ be the standard charts. On $U_i$, take the defining function

$$f_i(x^1, \ldots, \hat{x}^i, \ldots, x^{n+1}) = F(x^1, \ldots, 1, \ldots, x^{n+1}),$$

which vanishes exactly on $U_i \cap S$. To check the Jacobian criterion, note that since $x^j = \frac{X^j}{X^i}$ for $j \neq i$, we have

$$\frac{\partial f}{\partial x^j} = \frac{\partial F}{\partial X^j}.$$

If any of these are nonvanishing at $p$, we are done. On the other hand, if all of them vanish, by Euler's formula, we have

$$0 = F(x^1, \ldots, 1, \ldots, x^{n+1}) = \sum_{j \neq i} x^j \left. \frac{\partial F}{\partial X^j} \right|_p + \left. \frac{\partial F}{\partial X^i} \right|_p = \left. \frac{\partial F}{\partial X^i} \right|_p,$$

so the last one also vanishes. But this contradicts our assumption that all the partials do not vanish simultaneously on $S$. ◻

**Example 14.11.** We return to Example 14.6 yet again. We described in Example 14.8 how to extend the ambient domain from $\mathbb{R}^2$ to $\mathbb{C}^2$ to obtain a surface. We can extend further to $\mathbb{CP}^2$ just by writing down the homogeneous polynomial

$$F_\lambda(X, Y, Z) = ZY^2 - X\left(X - Z\right)\left(X - \lambda Z\right).$$

We have $F_\lambda(x, y, 1) = f_\lambda(x, y)$, so the zero locus of $F_\lambda$ agrees with $S_\lambda$ when restricted to the standard coordinate chart. You'll check on homework that $F_\lambda$ satisfies the Jacobian criterion, so defines a smooth curve in $\mathbb{CP}^2$. This curve is now compact, since it is a closed subset of a compact space. We have *compactified* the affine curve to a projective curve.

More generally, one can compactify a curve in $\mathbb{C}^2$ by defining the homogeneous polynomial $F(X,Y,Z) = Z^d f(\frac{X}{Z},\frac{Y}{Z})$. This may or may not satisfy the Jacobian criterion on the line at infinity $\{[X,Y,0]\}$. If it does, one obtains a smooth closed surface. A little bit more work is required to show that this surface is orientable.

Meanwhile, we know that compact surfaces are classified by genus. A very interesting (classical) question is: given a homogeneous polynomial of degree $d$ satisfying the Jacobian criterion, *what is the genus of the resulting curve in* $\mathbb{CP}^2$*?* The answer turns out to be:

$$g = \frac{(d-1)(d-2)}{2}.$$

Although it will be possible to prove this with the tools of Math 761, the result belongs more properly to a course on Riemann surfaces.

## 15. The embedding problem (Mon 10/7)

Today we will discuss the following classical

**Question.** Which manifolds are diffeomorphic to submanifolds of $\mathbb{R}^N$, for some (possibly large) $N$?

*Answer.* All of them.

We will prove the compact case and state a more precise general result (known as the *Whitney embedding theorem*) at the end of the section.

The embedding theorem itself is much less important and interesting than the idea of the proof, which relies on Sard's theorem. That is a cornerstone of the field of *differential topology*. However, this really requires a course of its own. Since we are instead headed towards differential *geometry* next semester, it probably makes more sense to prioritize other topics.

From the geometric point of view, it definitely is interesting to see some more explicit examples of embeddings. We can also give a nifty application (Theorem 15.3) which does not follow from the abstract embedding theorem.

As of now, we know how to embed $S^n$ (obviously), $T^2$ (Example 11.8), higher-genus orientable surfaces (by making a drawing), and $\mathbb{RP}^2$ (on homework) in Euclidean space. In class we looked at the immersed Klein bottle in $\mathbb{R}^3$ and convinced ourselves that just by adding a dimension we could easily make it an embedding. Let's see if we can also do projective spaces and Grassmannians.

**Example 15.1.** Define $F : K\mathbb{P}^n \to K^{(n+1)^2}$ by

$$\left[X^1,\dots,X^{n+1}\right] \mapsto \left(\frac{X^i\bar{X}^j}{|X|^2}\right)_{i,j=1}^{n+1}.$$

This map is well-defined and smooth by Corollary 12.8. To see that it's injective, let $X$ and $Y$ with $|X| = |Y| = 1$ and assume that $F(X) = F(Y)$. We then have

$$X^i \bar{X}^j = Y^i \bar{Y}^j$$

for all $i, j$. Considering the case $i = j$, we have

$$|X^i|^2 = |Y^i|^2$$

for each $i$. Now, choose any $j$ such that $X^j \neq 0$, and divide the above equation by this one, to obtain

$$X^i \frac{\bar{X}^j}{|X^j|^2} = Y^i \frac{\bar{Y}^j}{|Y^j|^2}$$
$$\frac{X^i}{X^j} = \frac{Y^i}{Y^j}.$$

Letting $\lambda = \frac{X^j}{Y^j}$, this translates to

$$X^i = \lambda Y^i$$

for all $i$. So $[X] = [Y]$, as desired. Since $K\mathbb{P}^n$ is compact, $F$ is a homeomorphism onto its image. One can check that the differential is injective using a slightly longer calculation than the above. However, we will instead generalize this construction to Grassmannians and check it there.

Notice that the image of this map actually lies in the unit sphere inside $K^{(n+1)^2}$, because

$$|F(X)|^2 = \sum_{i,j} \frac{X^i \bar{X}^j \left(\bar{X}^i X^j\right)}{|X|^4} = \sum_{i,j} \frac{|X^i|^2 |X^j|^2}{|X|^4} = 1.$$

**Example 15.2.** We will show that the Grassmannian $\mathrm{Gr}_k(K^n)$ also embeds into $K^{(n+1)^2}$. Given a plane $X \in \mathrm{Gr}_k(K^n)$, we can define the orthogonal projection:

$$P_X : K^n \to X$$
$$v \mapsto \sum_{i=1}^{k} \langle v, e_i \rangle e_i,$$

where $\{e_i\}$ is any orthonormal basis for $X$. This is the unique linear endomorphism of $K^n$ which satisfies

$$P_X^2 = P_X, \quad P_X^* = P_X, \quad \operatorname{Im} P_X = X. \tag{15.1}$$

So we can define the required map by sending

$$X \mapsto P_X,$$

which is injective. We will now argue that this is an embedding.

Suppose that the plane $X$ is spanned by an $n \times k$ matrix of full rank, which we also call $X$ (although this matrix is not unique). If the columns of $X$ are orthonormal, then we have the standard formula $P_X = XX^*$. The following formula works for any matrix with linearly independent columns:

$$P_X = X \left(X^* X\right)^{-1} X^*.$$

The matrices in the product are of dimensions $n \times k$, $k \times k$, and $k \times n$, respectively, so the product is $n \times n$. It is easy to check that this satisfies the requirements (15.1).

Now let $X_0$ be an arbitrary point in the Grassmannian. We can choose a standard coordinate chart $U$ with $Q = X_0^\perp$ and $P = X_0$, per Example 4.6. A point $X \in U$ is represented by a matrix

$$X = \begin{pmatrix} \mathbf{1} \\ A \end{pmatrix},$$

where $A \in \mathrm{Mat}_K^{(n-k)\times k}$ is arbitrary. We have

$$X_0 = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}.$$

From the above formula, we have

$$P_X = \begin{pmatrix} (1 + A^* A)^{-1} & (1 + A^* A)^{-1} A^* \\ A (1 + A^* A)^{-1} & A (1 + A^* A)^{-1} A^* \end{pmatrix}.$$

Since $1 + A^* A$ is positive-definite, it is invertible, so the expression makes sense on $U$ and gives a smooth function. The derivative at $X_0$ is simply

$$dF_{X_0}(Z) = \begin{pmatrix} 0 & Z^* \\ Z & 0 \end{pmatrix},$$

which is injective. An inverse map is given by the restriction to $\mathrm{Im}\, F$ of the map

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{1} \\ DB^{-1} \end{pmatrix},$$

which is smooth on an open set containing the image. We conclude that $F$ is an embedding.

**Theorem 15.3.** *The Grassmannian* $\mathrm{Gr}_k(K^n)$ *is compact.*

*Proof.* We have demonstrated that $\mathrm{Gr}_k(K^n)$ is homeomorphic to the set of orthogonal projection operators on $K^n$ of rank $k$. Notice that the rank of a projection operator is simply the trace. So this subset is defined by the three conditions

$$P^2 = P, \quad P^* = P, \quad \mathrm{Tr}\, P = k. \tag{15.2}$$

These are each polynomial (or linear) in the matrix entries, so define closed subsets of $\mathrm{Mat}_K^{n\times k}$, whose intersection is again closed.

Note that the set of such $P$ is also bounded, because

$$|P|^2 = \mathrm{Tr}\, PP^* = \mathrm{Tr}\, P^2 = \mathrm{Tr}\, P = k.$$

Therefore it is contained in the sphere of radius $\sqrt{k}$ inside $\mathrm{Mat}_K^{n\times k}$ (compare with the last example!). By Heine-Borel, the set defined by (15.2) is compact; so is $\mathrm{Gr}_k(K^n)$. □

We did not discuss the next result in class due to lack of time, but we include it here because the proof is so simple. (It might be called the *very weak* Whitney embedding theorem.)

**Proposition 15.4.** *Given a compact manifold* $M$, *there exists* $N \in \mathbb{N}$ *such that* $M$ *can be embedded in* $\mathbb{R}^N$.

*Proof.* Cover $M$ with $L$ regular coordinate balls $B_i$, such that $\overline{B_i} \subset B_i'$ and the chart $\varphi_i$ maps from $B_i'$ to $\mathbb{R}^n$. Let $\rho_i$ be a smooth bump function for $B_i \subset B_i'$. Define

$$F(p) = (\rho_1\varphi_1(p), \dots, \rho_L\varphi_L(p), \rho_1(p), \dots, \rho_L(p)) .$$

This is injective: if $F(p) = F(q)$ then for $p \in B_i$, we have $1 = \rho_i(p) = \rho_i(q)$. This implies that $p, q \in B_i'$, so we must have $\varphi_i(p) = \varphi_i(q)$. But $\varphi_i$ is injective on $B_i'$, so $p = q$.

To see that $F$ is an immersion, let $p \in B_i$. Then $(d\varphi_i)_p$ is injective, and since this is a component of $dF_p$, the latter is also injective.

Since $M$ is compact, we have a homeomorphism onto $F(M) \subset \mathbb{R}^{L(n+1)}$. □

**Theorem 15.5** (Whitney embedding/immersion). *A smooth manifold of dimension $n$ can be embedded in $\mathbb{R}^{2n}$ and immersed in $\mathbb{R}^{2n-1}$.*

*Proof.* The proof of the same statement with $2n+1$ in place of $2n$, sometimes called the "weak" version, is in Lee. The strong version requires another trick discovered by Whitney. All of these results start from the embedding of the previous proposition (or the version for manifolds with boundary) and proceed by playing with it to reduce the ambient dimension. □

# Part 4. Vector fields

## 16. Vector fields, derivations, and the Lie bracket (Wed 10/09)

### 16.1. Equivalent definitions of smoothness.

**Definition 16.1.** Let $\pi : T \to X$ be any map of sets. Let $U \subset X$ be a subset. A **section** of $\pi$ over $U$ is a map $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{Id}_U$. In other words,

$$\sigma(x) \in \pi^{-1}(x)$$

for all $x \in U$.

If $T$ and $X$ are topological spaces, resp. smooth manifolds, then we will assume that $\pi$ is continuous, resp. smooth, and the set $U$ is open. Then we can talk about continuous or smooth sections over $U$.

Recall that we defined the *tangent bundle*

$$TM = \sqcup_{x \in M} T_x M$$

as a set, and used the smooth manifold chart lemma to give it a topological and smooth structure. This was defined by the charts

$$\tilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$$

which were defined by

$$(x, v) \mapsto (\varphi_\alpha(x), (v^1, \ldots, v^n)),$$

where $v = \sum_i v^i \frac{\partial}{\partial x^i}$ in the coordinates defined by $\varphi_\alpha$. Here $\pi : TM \to M$ is the projection map taking all of $T_x M$ to $x$.

**Definition 16.2.** A **rough** vector field over $U$ is just a set-theoretic section $X : U \to TM|_U$. A **smooth** vector field is a smooth section. We let $\mathscr{X}(U)$ denote the vector space of smooth vector fields over $U$.

Note that a rough vector field $X$ defines a map

$$\begin{aligned} X : C^\infty(U) &\to \mathrm{Fun}(U, \mathbb{R}) \\ X(f)(p) &= X_p(f) \end{aligned} \tag{16.1}$$

This is a derivation, meaning that

$$X(fg) = X(f)g + fX(g)$$

where the equality is in the sense of functions, i.e., it holds for every $p \in U$.

**Proposition 16.3.** *Let $X$ be a rough vector field over $U$. The following are equivalent:*

1. *$X$ is smooth (i.e. $X \in \mathscr{X}(U)$).*
2. *In any coordinate chart $V$ meeting $U$, we have*

$$X|_{U \cap V} = \sum X^i(x) \frac{\partial}{\partial x^i},$$

*where the* **component functions** *$X^i(x)$ are smooth on $U \cap V$.*

3. *The image of the map (16.1) lies in* $C^\infty(U) \subset \mathrm{Fun}(U, \mathbb{R})$*, i.e.,* $X$ *is a derivation on the ring* $C^\infty(U)$.

*Proof.* Items (1) and (2) are equivalent by the definition of the smooth structure on $M$.

That (1-2) imply (3) is also clear by the local expression.

To go from (3) to (2), define the smooth functions $X^i(x) = X(x^i)$. Note that $f \mapsto X(f)(p)$ is a derivation at $p$. We showed on homework that every derivation at $p$ is a sum of partials; in particular, $X(f)(p) = \sum X^i(p) \left.\frac{\partial f}{\partial x^i}\right|_p$ at each point $p \in U$. Since the two are equal at each point, they are equal on functions. □

**Corollary 16.4.** *Every smooth derivation* $X : C^\infty(U) \to C^\infty(U)$ *corresponds to a smooth vector field.*

*Proof.* Given a derivation $X$, one obtains a rough vector field by letting $X_p(f) = X(f)(p)$, which is a derivation at $p$ by definition. Then apply the previous proposition. □

16.2. **Algebraic digression.** Let $\mathcal{R}$ be a commutative algebra over a field $K$. Throughout this section, $X, Y$, and $Z$ will denote three arbitrary derivations on $\mathcal{R}$, i.e. $K$-linear endomorphisms satisfying the Leibniz rule. We will later go back to the case of vector fields acting by derivations on $\mathcal{R} = C^\infty(U)$, but it is useful to prove a few statements without this baggage.

Define the **commutator bracket**

$$[X, Y] = XY - YX,$$

meaning, the endomorphism whose value on $f \in \mathcal{R}$ is given by $[X,Y](f) = X(Y(f)) - Y(X(f))$.

**Lemma 16.5.** $[X, Y]$ *is again a derivation.*

*Proof.* $K$-linearity is obvious. To check the Leibniz rule, for $f, g \in \mathcal{R}$, we calculate

$$\begin{aligned}[X,Y](fg) &= X(Y(f))g + X(f)Y(g) + X(g)Y(f) + fX(Y(g)) \\ &\quad - Y(X(f))g - Y(f)X(g) - Y(g)X(f) - fY(X(g)).\end{aligned}$$

The cross-terms cancel, and we are left with

$$[X,Y](fg) = [X,Y](f)g + f[X,Y](g),$$

as claimed. □

The following Lemma does not require $X, Y, Z$ to be derivations, but holds for any endomorphisms (or more generally, elements of an associative algebra).

**Lemma 16.6.** *The commutator bracket satisfies*

- $[X, Y] = -[Y, X]$ *(antisymmetry)*
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(Jacobi identity).*

*Proof.* The antisymmetry is obvious from the definition. To prove the Jacobi identity, we calculate:

$$\begin{aligned} LHS &= XYZ - XZY - YZX + ZYX \\ &\quad + YZX - ZYX - ZXY + YXZ \\ &\quad + ZXY - YXZ - XYZ + XZY. \end{aligned} \tag{16.2}$$

The terms cancel in pairs. □

**Remark 16.7.** We will reinterpret the Jacobi identity at least once below, perhaps making it seem less like an algebraic coincidence.

**Definition 16.8.** A **Lie algebra** is a vector space $V$ together with a binary operation $[-,-] : V \times V \to V$ which is bilinear, antisymmetric, and satisfies the Jacobi identity.

**Example 16.9.** We have shown that any associative algebra over $K$, equipped with the commutator bracket, is a Lie algebra. In particular, the algebra of endomorphisms on any vector space (under composition, which is associative) is a Lie algebra.[6] Finally, we have shown that the space of derivations on a commutative algebra forms a Lie subalgebra of the space of all endomorphisms of the underlying vector space.

**Definition 16.10.** A **derivation** on a Lie algebra $V$ is an endomorphism $D : V \to V$ that satisfies

$$D[Y,Z] = [DY,Z] + [Y,DZ]$$

for all $Y, Z \in V$.

**Lemma 16.11.** *Let $X \in V$ be an element of a Lie algebra. Define the endomorphism $D_X : V \to V$ by*

$$D_X Y = [X,Y].$$

*Then $D_X$ is a derivation.*

*Proof.* Rearranging the second a third terms of the Jacobi identity using antisymmetry of the bracket, we have

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]. \tag{16.3}$$

This is precisely the statement that $D_X$ is a derivation on the Lie algebra $V$. □

**Remark 16.12.** This gives perhaps a more satisfying formulation to the Jacobi identity. It also suggests that the bracket $[X,-]$ represents some sort of derivative. Later we will see this geometrically in the case of vector fields.

[6]In fact, it is a theorem (Ado's Theorem) that every finite-dimensional Lie algebra over $K$ arises in this way.

16.3. **The Lie bracket of vector fields.** We showed above that smooth vector fields $X \in \mathscr{X}(U)$ are equivalent to derivations on $C^\infty(U)$. By Lemma 16.5, we may make the following definition:

**Definition 16.13.** The **Lie bracket** of two smooth vector fields $X, Y \in \mathscr{X}(U)$ is the derivation $[X,Y] \in \mathscr{X}(U)$. This acts on a smooth function $f \in C^\infty(U)$ by

$$[X,Y](f) = X(Y(f)) - Y(X(f)).$$

Working in a local coordinate system $\{x^i\}_{i=1}^n$, we have $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$ for smooth functions $X^i(x)$ and $Y^i(x)$ (the local components). The Lie bracket is then given by

$$\begin{aligned}[X,Y](f) &= \sum_{i,j} \left[ X^i \frac{\partial}{\partial x^i}, Y^i \frac{\partial}{\partial x^i} \right] \\ &= \sum_{i,j} X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} \\ &\quad - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} - X^i Y^j \frac{\partial^2 f}{\partial x^j \partial x^i}.\end{aligned}$$

Since partials commute, the last terms cancel. If we switch the indices $i$ and $j$ on the first term, factor out $\frac{\partial f}{\partial x^i}$, and ignore $f$, we can write

$$[X,Y] = \sum_{i,j} \left( X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

The $i$'th component of $[X,Y]$ is given by

$$\begin{aligned}[X,Y]^i &= \sum_j \left( X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \\ &= X(Y^i) - Y(X^i).\end{aligned} \tag{16.4}$$

**Example 16.14.** Let $X = \frac{\partial}{\partial x^{j_1}}$ and $Y = \frac{\partial}{\partial x^{j_2}}$ be coordinate vector fields. Since partials commute, or by the above formula with $X^i = \delta^i{}_{j_1}$ and $Y^i = \delta^i{}_{j_2}$, we see that $[X,Y] = 0$.

**Example 16.15.** Suppose $M = \mathbb{R}^n$ and consider the vector field with components $X^i = x^i$:

$$X = \sum_i x^i \frac{\partial}{\partial x^i}.$$

This is called the **radial vector field** or sometimes the *Euler vector field* for reasons we'll see shortly.

Let $Y^i(x)$, $i = 1, \dots, n$, be a collection of homogeneous polynomials of the same degree, $d$, and take $Y = \sum Y^i(x) \frac{\partial}{\partial x^i}$. We then have

$$\begin{aligned}[X,Y]^i &= \sum_j x^j \frac{\partial Y^i}{\partial x^j} - Y^j \delta^i{}_j \\ &= \sum_j x^j \frac{\partial Y^i}{\partial x^j} - Y^i.\end{aligned}$$

By Euler's formula, Lemma 14.9, we have

$$[X,Y] = (d-1)Y.$$

We will explain this result later based on the fact that $X$ is the generator of homothetic rescaling on $\mathbb{R}^n$.

## 17. Vector fields tangent to submanifolds, $F$-related vector fields (Fri 10/11)

### 17.1. Vector fields tangent to submanifolds.

**Definition 17.1.** Let $S \subset M$ be a submanifold. A vector field $X$ on $M$ is **tangent** to $S$ if $X_p \in T_pS \subset T_pM$ for all $p \in S$.

**Proposition 17.2.** *TFAE:*

1. *$X$ is tangent to $S$*
2. *$X$ annihilates any collection of defining functions on $S$, i.e., if $f^1, \dots, f^{n-k}$ are defining functions then $X(f^i)|_S = 0$ for $i = 1, \dots, k$.*
3. *For any function $f$ that is constant on $S$, $X(f)|_S \equiv 0$.*

*Proof.* Given a set of defining functions, we can choose a slice chart $\{x^1, \dots, x^k, y^1, \dots, y^{n-k}\}$ such that $y^i = f^i$ for $i = 1, \dots, n-k$. Writing

$$X = \sum_{i=1}^{k} X^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{n-k} X^{k+j} \frac{\partial}{\partial y^j},$$

we have $X$ tangent to $S$ if and only if $X^{k+j}(x^1, \dots, x^k, 0, \dots, 0) \equiv 0$ for $j = 1, \dots, n-k$.

To see (1) $\iff$ (2), note that $X(y^j) = X^{k+j}$, so these vanish on $S$ if and only if $X$ is tangent to $S$.

(3) $\Rightarrow$ (2) is clear. So see (1) $\Rightarrow$ (3), note that if $f$ is constant on $S$ then $\frac{\partial f}{\partial x^i} \equiv 0$ on $S$ for $i = 1, \dots, k$. Since also $X^{k+j} \equiv 0$ on $S$, we have $X(f) \equiv 0$ on $S$, as claimed. □

**Corollary 17.3.** *Suppose $X$ and $Y$ are both tangent to $S$. Then $[X,Y]$ is also tangent to $S$, and*

$$[X,Y]|_S = [X|_S, Y|_S].$$

*Proof.* By (3) of the previous Lemma, we have $Y(f) = X(f) = 0$ on $S$ if $f$ is constant on $S$. In particular, these are constant, so we also have $X(Y(f)) = Y(X(f)) = 0$ on $S$. Therefore $[X,Y]f = 0$ on $S$, so again by (3), $[X,Y]$ is tangent to $S$.

To see that the Lie bracket on $S$ is the restriction of the Lie bracket on $M$, we calculate in a slice chart as before. For $1 \leq i \leq k$, the $i$'th component of $[X,Y]$ is given by

$$\begin{aligned} [X,Y]^i\big|_S &= \sum_{j=1}^{n} \left( X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right)\Bigg|_S \\ &= \sum_{j=1}^{k} \left( X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right)\Bigg|_S \\ &= [X|_S, Y|_S]. \end{aligned}$$

□

**Example 17.4.** To check that a vector field on $\mathbb{R}^n$ is tangent to $S^{n-1}$, intuitively, we should check that it is orthogonal to the radial vector field. Let's prove that this is correct. A defining function for $S^n$ is

$$|x|^2 = \sum (x^i)^2 = 1.$$

By the previous Lemma, a vector field $X$ is tangent to $S^{n-1}$ if and only if it annihilates the defining function. We have:

$$X(|x|^2) = 2\sum X(x^i)x^i = 2\sum X^i x^i = 2\langle X, x\rangle.$$

So indeed $X$ is tangent iff it is orthogonal to the radial vector field $x$.

**Example 17.5.** Let $M = \mathbb{R}^{2n}$. Define the vector field

$$X = \sum_{i=1}^{n}\left(x^{2i}\frac{\partial}{\partial x^{2i-1}} - x^{2i-1}\frac{\partial}{\partial x^{2i}}\right).$$

We claim that this restricts to a *nowhere-vanishing* vector field on $S^{2n-1}$. First we must check that it is tangent:

$$\langle X, x\rangle = \sum x^{2i}x^{2i-1} - x^{2i-1}x^{2i} = 0.$$

It vanishes nowhere on $S^n$ because the same is true of $X$ on $\mathbb{R}^n$, by examining its coefficients.

Note that if we include $\mathbb{R}^{2n} \subset \mathbb{R}^{2n+1}$ and use the same expression to define a vector field, we obtain one that vanishes only on the $x^{2n+1}$-axis, so only at two points of $S^n$. This is the one you may have used on your homework. (We will see shortly that these indeed descend to projective space).

**Example 17.6.** Consider the following three vector fields on $S^2 \subset \mathbb{R}^3$ :

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

You will show on homework that these satisfy

$$[X, Y] = Z$$

plus cyclic permutations. By Corollary 17.3, you can work on $\mathbb{R}^3$, so this is just a simple calculation. The closure under taking brackets implies that $X, Y$, and $Z$ span a 3-dimensional Lie subalgebra of $\mathscr{X}(S^2)$. Later we will understand why this is the case.

17.2. **$F$-related vector fields.** Recall that we can push forward a tangent vector by a smooth map $F$, either by post-composing paths with $F$ (in $T_x^{(4)}M$) or at the level of derivations (in $T_x^{(2)}M$):

$$dF_p(v) : f \mapsto v(f \circ F).$$

Note that we **cannot in general** push forward an entire vector field, for the obvious reason that if $p, p' \in F^{-1}(q)$ then we may have $dF_p(X_p) \neq dF_{p'}(X_{p'})$, in which case the pushforward of $X$ would simply not be well-defined. (There is also the question of smoothness...see below.) Rather, we will work with the following notion.

**Definition 17.7.** Let $F : M \to N$ be smooth, $X \in \mathscr{X}(M), Y \in \mathscr{X}(N)$. We say that $X$ and $Y$ are $F$-related if whenever $q = F(p)$, we have

$$dF_p(X_p) = Y_q.$$

**Example 17.8.** Given a submanifold $S \subset M$, let $\iota$ denote the inclusion map, which is an embedding. Supposing that $X$ is tangent to $S$, the vector fields $X|_S$ and $X$ are $\iota$-related.

We will now generalize Corollary 17.3 to the situation of general $F$-related vector fields.

**Lemma 17.9.** *Two vector fields $X$ and $Y$ are $F$-related if and only if for all $f \in C^\infty(U)$, $U \subset N$, we have*

$$X(f \circ F) = Y(f) \circ F.$$

*Proof.* Given $p \in M$, we have

$$LHS(p) = X_p(f \circ F) = dF_p(X_p)(f).$$

$$RHS(p) = (Y(f))(F(p)) = Y_{F(p)}(f).$$

These are equal for all $f$ if and only if $dF_p(X_p) = Y_{F(p)}$ for all $p \in M$. □

**Proposition 17.10.** *Suppose $X_1$ and $X_2$ are $F$-related to $Y_1$ and $Y_2$, respectively. Then $[X_1, X_2]$ is $F$-related to $[Y_1, Y_2]$.*

*Proof.* Applying the Lemma twice, we have

$$X_1(X_2(f \circ F)) = X_1(Y_2(f) \circ F) = Y_1(Y_2(f)) \circ F.$$

Similarly, $X_2(X_1(f \circ F)) = Y_1(Y_2(f)) \circ F$. Subtracting, we get

$$[X_1, X_2](f \circ F) = ([Y_1, Y_2](f)) \circ F.$$

Since $f$ was arbitrary, the other direction of the Lemma implies the claim. □

## 18. Descent of vector fields, quotient manifolds (Fri 10/11)

We will now describe a few situations where you *can* safely push forward a vector field. This can be handy for constructing vector fields on quotient manifolds.

**Proposition 18.1** (Smooth descent by submersions). *Suppose that $F : M \to N$ is a surjective*[7] *submersion. If $X \in \mathscr{X}(M)$ is such that $dF_p(X_p)$ is constant on fibers (i.e. for all $q \in N$, $dF_p(X_p) = dF_{p'}(X_{p'})$ for all $p \in F^{-1}(q)$), then there exists $Y \in \mathscr{X}(N)$ (the "pushforward") that is $F$-related to $X$.*

[7] If $F$ is not surjective, one can simply replace $N$ by the image of $M$, which is an open subset because submersions are open maps.

*Proof.* We can define a rough vector field $Y$ on $N$ by

$$Y_q = dF_p(X_p)$$

for any $p \in F^{-1}(q)$; this is well-defined by assumption. We can then check smoothness in any coordinate chart. Since $F$ is a submersion, there exist charts centered at $p$ and $q$ such that $F$ takes the form

$$F(x^1, \dots, x^n, y^1, \dots, y^{m-n}) = (x^1, \dots, x^n).$$

By definition, for $1 \le i \le n$, we have

$$Y^i(x^1, \dots, x^n) = X^i(x^1, \dots, x^n, 0, \dots, 0),$$

which is smooth by assumption. ☐

**Corollary 18.2.** *One can always push forward a vector field by a diffeomorphism. In particular, the pushforward $F_*X$ of $X$ is given by*

$$(F_*X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}). \tag{18.1}$$

*Proof.* Since $F$ is a diffeomorphism, it is a submersion, and the well-definedness is automatic since there is only one point in each fiber. The formula is gotten by taking $p = F^{-1}(q)$ in the definition of $F$-relatedness. ☐

We now combine the previous two results into the form in which they are most commonly used. It is convenient to make the following definition now; we will only discuss it more thoroughly later on.

**Definition 18.3.** Suppose that a group $G$ acts by diffeomorphisms on a smooth manifold, $M$. We say that $N$ is the **quotient manifold** of $M$ by $G$, and write $N = M/G$, if there exists a surjective submersion $F : M \to N$ such that the fibers of $F$ are equal to the orbits of $G$, and in the diagram

$$\begin{array}{ccc} M & & \\ \downarrow & \searrow^{F} & \\ M/G & \xrightarrow{\sim} & N, \end{array}$$

the bottom map is a homeomorphism.

**Remark 18.4.** A few comments on the previous definition.

- The condition that the fibers of $F$ are equal to the orbits of $G$ is really two conditions: that $G$ preserves the fibers (i.e. $F(gx) = F(x)$ for all $x \in M, g \in G$) and acts transitively on them (i.e. $Gx = F^{-1}(F(x))$ for all $x \in M$).
- It follows by Theorem 12.10, already proven above, that $N$ is unique up to diffeomorphism, if it exists.
- Later, we plan to give some fairly general conditions on the group action guaranteeing the existence of a smooth quotient manifold $N$. You have already shown on homework that if the action is properly discontinuous then $M/G$ has a smooth structure.

**Proposition 18.5.** *Let $N = M/G$ be a quotient manifold as above, with $F : M \to N$ the (smooth) quotient projection. Suppose that $X$ is a $G$-invariant vector field, i.e., $g_* X = X$ for all $g \in G$. Then $X$ descends to a smooth vector field $Y$ on the quotient that is $F$-related to $X$.*

*Proof.* Since in the definition of quotient manifold we assume that $F$ is a surjective submersion, it remains only to check the well-definedness. Let $p, p' \in F^{-1}(q)$. Since $G$ acts transitively on fibers, we have $p' = g(p)$ for some $g \in G$. Since $X$ is $G$-invariant, we have

$$X_{p'} = X_{g(p)} = dg_p(X_p).$$

Since $g$ preserves fibers, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\;g\;} & M \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle F} & \\ & N & \end{array}$$

commutes. We therefore have

$$dF_p = dF_{g(p)} \circ dg_p.$$

Applying $dF_{p'}$ to the previous, we get

$$dF_{p'}(X_{p'}) = dF_{p'}(dg_p(X_p)) = dF_p(X_p),$$

as desired. □

We'll now describe two ways to use this proposition to write down vector fields on $\mathbb{RP}^n$.

**Example 18.6.** Think of $\mathbb{RP}^n = S^n / \pm 1$. Given a vector field $X$ on $\mathbb{R}^{n+1}$ that is tangent to $S^n$ and all of whose coefficients are *odd*, i.e.

$$X^i(-x) = -X^i(x),$$

we claim that $X$ descends to a vector field on $\mathbb{RP}^n$. Let $\alpha : x \to -x$ be the antipodal map on $\mathbb{R}^{n+1}$, whose differential is

$$d\alpha = -\mathrm{Id}$$

at all points. By formula (18.1), we have

$$\begin{aligned} (\alpha_* X)_x &= d\alpha_{-x}\left(X_{-x}\right) \\ &= -\sum X^i(-x) \frac{\partial}{\partial x^i} \\ &= X, \end{aligned}$$

since the coefficients are odd functions. Since $X$ is invariant under the antipodal map, it descends.

This construction has the advantage that since the projection is a local diffeomorphism, the projected vector field vanishes at $[p]$ if and only if $X$ vanishes at $p$.

**Example 18.7.** Think of $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times$, where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ acts by scalar multiplication. Given a vector field $X$ on $\mathbb{R}^{n+1}$ all of whose coefficients are *linear*, i.e. homogeneous of order one, we claim that $X$ descends to a vector field on $\mathbb{RP}^n$. (Note that $X$ does not need to be tangent to $S^n$.)

Let $\alpha_\lambda : x \mapsto \lambda x$ denote scalar multiplication. We have

$$d\alpha_\lambda = \lambda \mathrm{Id}$$

at all points. By (18.1), we have

$$\begin{aligned}((\alpha_\lambda)_* X)_x &= d(\alpha_\lambda)_{\lambda^{-1}x}(X_{\lambda^{-1}x}) \\ &= \lambda \sum X^i(\lambda^{-1}x)\frac{\partial}{\partial x^i} \\ &= X,\end{aligned}$$

since the coefficients are linear functions. Hence $X$ descends.

This construction has the advantage that it works over $\mathbb{C}$ and gives *holomorphic* vector fields on $\mathbb{CP}^n$. That discussion is for another class, however.

## 19. Integral curves and flows (Mon 10/14)

19.1. **Integral curves.** Recall that a tangent vector at $p$ is equivalent to the derivative of a path through $p$. Today we will describe the relationship between vector *fields* and paths.

**Definition 19.1.** Let $X \in \mathscr{X}(M)$. Let $J \subset \mathbb{R}$ be an open interval. A path $\gamma : J \to M$ is said to be an **integral curve** of $X$ if

$$\gamma'(t) = X_{\gamma(t)}$$

for all $t \in J$.

In a coordinate chart where $\gamma(t) = (x^1(t), \dots, x^n(t))^T$ and $X = \sum X^i \frac{\partial}{\partial x^i}$, we have $\gamma'(t) = \sum \frac{dx^i}{dt}\frac{\partial}{\partial x^i}$, so the condition that $\gamma$ is an integral curve amounts to

$$\begin{aligned}\frac{dx^1}{dt} &= X^1(x^1(t), \dots, x^n(t)) \\ &\vdots \\ \frac{dx^n}{dt} &= X^n(x^1(t), \dots, x^n(t)).\end{aligned} \tag{19.1}$$

This is a system of ordinary differential equations in $n$ variables. We have the following classical result:

**Theorem 19.2** (Picard's local existence and uniqueness theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open set, and suppose given a set of smooth functions $X^1, \dots, X^n$ on $U$. Given $p \in U$, there exists $\varepsilon > 0$ and $V \ni p$ such that for all $q \in V$, there exists a* **unique** *solution $\gamma_q(t) = (x^1(t), \dots, x^n(t))$ of (19.1) with $\gamma(0) = q$ and $\gamma_q(t) \in U$ for $t \in (-\varepsilon, \varepsilon)$. Moreover, $\gamma_q(t) \in U$ depends smoothly on $(q, t) \in V \times (-\varepsilon, \varepsilon)$.*

*Proof.* If you haven't seen this result before, have a look at Lee, Appendix C. The theorem is proved by converting (19.1) to an integral equation and setting up a contraction mapping problem, similar to what we did to prove the Inverse Function Theorem, except in the Banach spaces $C^k((-\varepsilon, \varepsilon), \mathbb{R}^n)$. □

The uniqueness in Picard's theorem is crucial, because it allows us to globalize the result. In class I stated this as self-evident, here I will put it as a separate corollary.

**Corollary 19.3.** *Let $X \in \mathscr{X}(M)$. Given $p \in M$, there exists an open interval $I_p \subset \mathbb{R}$, containing zero, and unique* **maximal** *integral curve $\gamma_p : I_p \to M$ of $X$ with $\gamma_p(0) = p$. In other words, given any other integral curve $\tilde{\gamma} : J \to M$ of $X$ with $\tilde{\gamma}(0) = p$, we must have $J \subset I_p$ and $\tilde{\gamma}(t) = \gamma_p(t)$ for $t \in J$.*

*Proof.* First observe that given any two integral curves $\gamma_1(t)$ and $\gamma_2(t)$ on $I_1$ and $I_2$, respectively, with $\gamma_1(0) = \gamma_2(0) = p$, we must have $\gamma_1(t) = \gamma_2(t)$ on $I_1 \cap I_2$. This requires only a little bit of thought: let $J \subset I_1 \cap I_2$ be the set on which $\gamma_1 = \gamma_2$. Looking in a coordinate chart around $p$, by the uniqueness in Picard's theorem, we must have $J \ni 0$, so $J$ is nonempty. It is also open for the same reason, and closed in $I_1 \cap I_2$ since $\gamma_1(t)$ and $\gamma_2(t)$ are both continuous. Hence $J = I_1 \cap I_2$.

This allows us to define $\gamma_p(t)$ to the "union" of all integral curves through $p$, which is maximal by definition. □

Note that $I_p$ and $\gamma_p$ so defined clearly have the following properties:

$$\gamma_{\gamma_p(t)}(s) = \gamma_p(t+s) \tag{19.2}$$

$$I_{\gamma_p(t)} = I_p - t$$

Last, if $X$ and $Y$ are $F$-related, then $F \circ \gamma_p$ is an integral curve of $Y$ through $F(p)$. This follows from the chain rule.

By the theorem above, we know that integral curves always exist locally, but it is instructive to calculate some explicit examples.

**Example 19.4.** Take $M = \mathbb{R}^n$. Given any $n \times n$ matrix $A = (A^i{}_j)$, we can write down a so-called *linear vector field*

$$X = \sum_{i,j} A^i{}_j x^j \frac{\partial}{\partial x^i}.$$

The corresponding system of ODEs is

$$\frac{dx}{dt} = A \cdot x,$$

and we have

$$\gamma_p(t) = \exp(At) \cdot p$$

for every $p \in \mathbb{R}^n$. Thus $I_p = \mathbb{R}$ for all $p$ and each $\gamma_p(t)$ exists globally. (This is true more generally for vector fields/ODE systems on $\mathbb{R}^n$ with sub-linear growth.)

**Example 19.5.** For a specific case of the previous example, take

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding vector field is

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

The integral curve through $(x_0, y_0)^T$ is

$$\gamma_{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

We have

$$\gamma'_{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}(t) = \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \gamma_{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}(t)$$

as required. The integral curve travels around in a circle with constant velocity equal to the radius.

**Example 19.6.** On $M = \mathbb{R}$, take $X = x^2 \frac{\partial}{\partial x}$. The ODE $\frac{dx}{dt} = x^2$ can be solved by separation of variables, giving the integral curves

$$\gamma_x(t) = \frac{1}{x^{-1} - t}$$

for $x \neq 0$, and $\gamma_0(t) \equiv 0$. Notice that these are not globally defined, but rather

$$I_x = \begin{cases} (-\infty, x^{-1}) & x > 0 \\ \mathbb{R} & x = 0 \\ (x^{-1}, \infty) & x < 0. \end{cases}$$

19.2. **Flows.** We now change our perspective on the above. Let

$$\mathscr{D} = \cup_{p \in M} I_p \times \{p\} \subset \mathbb{R} \times M.$$

A union of intervals of this kind, each containing zero, is called a **flow domain**. We can define a map

$$\theta : \mathscr{D} \to M$$
$$\theta_t(p) = \gamma_p(t).$$

Notice that we have simply changed the order of the variables in the integral curves $\gamma_\cdot(\cdot)$. The map $\theta$ is called the **flow associated to** $X$, and satisfies the group-action-like properties

$$\theta_0 = \mathrm{Id}_M$$

and, from (19.2),

$$\theta_s \circ \theta_t = \theta_{s+t}$$

wherever the composition is defined. If $\mathscr{D} = \mathbb{R} \times M$, then the transformations $\theta_t$ do form a globally defined $\mathbb{R}$-action on $M$, called a *one-parameter family of diffeomorphisms*. We will give the general version of this statement below. Conversely, given $\theta$ of this form, $X = \frac{\partial \theta}{\partial t}\big|_{t=0}$ is called the *infinitesimal generator* of the flow $\theta$.

**Example 19.7.** In Example 19.5 above, we have

$$\theta_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos t - y \sin t \\ x \sin t + y \cos t \end{pmatrix}.$$

The flow is simply given by counterclockwise rotation on $\mathbb{R}^2$ by angle $t$.

The following summarizes the correspondence between vector fields and flows.

**Theorem 19.8** (Fundamental Theorem on Flows)**.** *Given $X \in \mathscr{X}(M)$, there exists an open set $\mathscr{D} \subset M$, containing $\{0\} \times M$, and a smooth map $\theta : \mathscr{D} \to \mathbb{R} \times M$, such that*

$$\mathscr{D} \cap (\mathbb{R} \times \{p\}) = I_p, \tag{19.3}$$

*the maximal interval of definition of $\gamma_p$ (see above), and*

$$\theta_t(p) = \gamma_p(t) \tag{19.4}$$

*for all $(t,p) \in \mathscr{D}$. Moreover, letting*

$$M_t = \mathscr{D} \cap (\{t\} \times M) \subset M, \tag{19.5}$$

*$\theta_t$ defines a diffeomorphism*

$$\theta_t : M_t \xrightarrow{\sim} M_{-t}. \tag{19.6}$$

*Proof.* We take (19.3-19.4) as our definition of $\theta$. It remains to check that $\mathscr{D}$ is open, $\theta$ is smooth on $\mathscr{D}$, and $\theta_t$ defines a diffeomorphism as in (19.6).

Let $W \subset \mathscr{D}$ be the set of all $(t,p) \in \mathscr{D}$ for which there exists an interval $J \subset \mathbb{R}$ containing 0 and $t$, and an open neighborhood $U \ni p$, such that $J \times U \subset \mathscr{D}$ and $\theta|_{J\times U}$ is smooth. Clearly $W$ is open in $\mathbb{R} \times M$. So it suffices to show that $W = \mathscr{D}$.

Suppose for the sake of contradiction that $W \subsetneq \mathscr{D}$, and let $(t_0, p_0) \in \mathscr{D} \setminus W$. First note that by Picard's Theorem, $\{0\} \times M \subset W$. Assume that $t_0 > 0$, since the other case is similar. Without loss of generality, we can assume that $t_0$ is the smallest positive value such that $(t_0, p_0) \in \mathscr{D} \setminus W$ (by openness of $W$).

Let $p = \theta_{t_0}(p_0) \in M$. Again by Picard's Theorem, there exists $V \ni p$ and $\varepsilon > 0$ such that the flow $\theta_s$ is smooth on $(-\varepsilon, \varepsilon) \times V$.

Meanwhile, since $(t_0 - \frac{\varepsilon}{2}, p_0) \in W$, there exists $J, U$ such that $\theta_t$ is smooth on $J \times U$, with $J \subset \left[0, t_0 - \frac{\varepsilon}{2}\right]$.

Define $U_0 = \theta_{t_0 - \frac{\varepsilon}{2}}\big|_U^{-1}(V) \subset U$, which is open. On $U_0 \times (0, t_0 + \varepsilon)$ we can write $\theta_t$ as a composition of smooth functions

$$\theta_t(p) = \theta_{t - t_0 + \varepsilon/2} \circ \theta_{t_0 - \varepsilon/2}(p).$$

But this shows that $(t_0, p_0) \in W$, which is a contradiction. We conclude that $W = \mathscr{D}$.

To prove (19.6), note that the image of the map $\theta_t : M_t \to M$ must be contained in $M_{-t}$, because any trajectory can be flown backwards to its initial point. In fact, $\theta_{-t} : M_{-t} \to M_t$ is a smooth inverse of $\theta_t$, so the map is a diffeomorphism. □

**Example 19.9.** In Example 19.6, where $X = x^2 \frac{\partial}{\partial x}$, we have

$$\mathscr{D} = \{(t,x) \mid tx < 1\}.$$

For $t > 0$, this gives

$$M_t = \left(-\infty, \frac{1}{t}\right)$$

and

$$M_{-t} = \left(-\frac{1}{t}, \infty\right).$$

So, according to the theorem, we have a diffeomorphism

$$\theta_t : \left(-\infty, \frac{1}{t}\right) \to \left(-\frac{1}{t}, \infty\right).$$

This is a rather interesting diffeomorphism, in that it crunches one side of the interval in from infinity while stretching the other side out to infinity. The formula is just

$$\theta_t(x) = \frac{x}{1 - tx}.$$

## 20. Properties of flows (Wed 10/16)

Here we will prove some simple facts about flows that may be needed later.

**Proposition 20.1.** *Let $X \in \mathscr{X}(M)$ and $Y \in \mathscr{X}(N)$ and suppose that $X$ and $Y$ are $F$-related. Let $\theta_t$ and $\eta_t$ be the flows of $X$ and $Y$, respectively. We have a commutative diagram:*

$$\begin{CD} M_t @>F>> N_t \\ @V{\theta_t}VV @VV{\eta_t}V \\ M_{-t} @>F>> N_{-t} \end{CD}\quad .$$

*Proof.* By the chain rule and $F$-relatedness, the path $t \mapsto (F \circ \theta_t)(p)$ is an integral curve of $Y$ starting at $F(p)$. On the other hand, so is $\eta_t(F(p)) = (\eta_t \circ F)(p)$, so the two must be equal. This shows both that the image of the top and bottom maps are contained in $N_t$ and $N_{-t}$, respectively, and that the diagram commutes. ☐

**Corollary 20.2.** *If $F : M \to M$ is a diffeomorphism, then the flow of $F_*X$ is given by $F \circ \theta_t \circ F^{-1}$.*

*Proof.* Let $\eta_t$ be the flow of $F_*X$, which is $F$-related to $X$ by definition. By the previous proposition, we have

$$\eta_t \circ F = F \circ \theta_t.$$

Precomposing both sides with $F^{-1}$ yields the result. ☐

**Definition 20.3.** A point $p \in M$ is called a **regular point** of $X$ if $X_p \neq 0$. If $X_p = 0$, $p$ is called a **singular point**.

Note that the terminology has nothing to do with smoothness of $X$, which we always assume.

Also note that $p$ is a singular point if and only if $\theta_t(p) \equiv p$ for all $t \in \mathbb{R}$, since this is the unique integral curve through $p$.

**Lemma 20.4.** *Suppose $p$ is a regular point of $X$. There exists a coordinate system $\{s^1, \dots, s^n\}$ near $p$ such that $X = \frac{\partial}{\partial s^1}$. Given any hypersurface $S$ passing through $p$ for which $X_p \notin T_pS$, we can further choose the coordinates so that $S = \{s^1 = 0\}$.*

*Proof.* Let $U = \{x^1, \dots, x^n\}$ be a slice chart for $S$, in which $S = \{x^1 = 0\}$. We may assume without loss of generality that $X$ does not vanish on $U$. Define a smooth map from a neighborhood $V$ of the origin in $\mathbb{R}^n$ to $U$ by

$$\Phi : (s^1, \dots, s^n) \mapsto \theta_{s^1}(0, s^2, \dots, s^n) \in U.$$

Since the curves $(t, s^2, \dots, s^n)$ are sent to integral curves of $X$, we have

$$d\Phi_p\left(\frac{\partial}{\partial s^1}\right) = X$$

for all $p \in V$. At the origin, for $i > 1$, we have

$$d\Phi_0\left(\frac{\partial}{\partial s^i}\right) = \frac{\partial}{\partial x^i}$$

by construction. Since $X$ and $\{\frac{\partial}{\partial x^i}\}_{i>1}$ are linearly independent at the origin, the inverse function theorem tells us that $\Phi$ is also a coordinate system. It has the required properties by construction. ☐

We now address the (thorny) question of the domain $\mathscr{D}$, which may prevent $\theta_t$ from giving a globally defined $\mathbb{R}$-action.

**Definition 20.5.** A vector field $X$ is **complete** if $\mathscr{D} = \mathbb{R} \times M$, or equivalently, if $I_p = \mathbb{R}$ for all $p \in M$ (i.e. every integral curve is complete).

**Lemma 20.6.** *Suppose $K \subset M$ is compact. There exists $\varepsilon > 0$ and an open set $U \supset K$ such that $(-\varepsilon, \varepsilon) \times U \subset \mathscr{D}$. In particular, $K \subset M_t$ for all $t \in (-\varepsilon, \varepsilon)$.*

*Proof.* Since $\mathscr{D}$ is open, we can cover $\{0\} \times K$ by a finite collection of neighborhoods $(-\varepsilon_i, \varepsilon_i) \times U_i$ (since these form a basis for the product topology on $\mathbb{R} \times M$). Let $\varepsilon = \min \varepsilon_i$ and $U = \cup U_i$. ☐

**Proposition 20.7.** *Suppose that $X \in \mathscr{X}(M)$ has compact support. Then $X$ is complete.*

*Proof.* Put $K = \operatorname{supp} X$ and let $\varepsilon > 0$ be as in the Lemma. We have $K \subset M_t$ for all $t \in (-\varepsilon, \varepsilon)$. Furthermore, since $X$ vanishes identically on $M \setminus K$, the latter is contained in $M_t$ for all $t \in \mathbb{R}$. We conclude that $M = M_t$ for all $t \in (-\varepsilon, \varepsilon)$.

Given $L \in \mathbb{R}_+$, choose $N$ large enough that $N\varepsilon > L$. For $s \in (-L, L)$, we can write

$$\theta_s = \overbrace{\theta_{s/N} \circ \cdots \circ \theta_{s/N}}^{N}.$$

This shows that $\theta_s$ is well-defined and smooth on $(-L, L) \times M$. Since $L$ was arbitrary, we are done. ☐

**Corollary 20.8.** *Every vector field on a compact manifold is complete.*

The flow construction therefore gives an easy way to produce global 1-parameter families of diffeomorphisms on compact manifolds.

On the other hand, the following confirms our intuition that an integral curve can be continued until it "runs off the manifold."

**Proposition 20.9.** *Suppose $M$ is noncompact. If $p \in M$ is a point such that $I_p = (a,b)$, with $b < \infty$, then the image $\gamma_p([0,b))$ fails to be contained in any compact subset of $M$.*

*Proof.* (Omitted during class.) We prove the contrapositive. Suppose that $\gamma$ is an integral curve of $X$ with $\gamma(0) = p$ and $\gamma([0,b)) \subset K$ for some compact set $K \subset M$. Choose a compact set $K'$ with $K \subset (K')^\circ$, and let $\varphi$ be a compactly supported bump function for $K'$. Take $\tilde{X} = \varphi X$, which agrees with $X$ on $K'$, and let $\tilde{\gamma} : \mathbb{R} \to M$ be the (complete) integral curve of $\tilde{X}$. We have $\gamma = \tilde{\gamma}$ on $[0,b)$ by uniqueness. But $\tilde{\gamma}(t)$ is also contained in $(K')^\circ$ for $t \in [b, b+\varepsilon)$, for some $\varepsilon > 0$, and on this interval it remains an integral curve of $X$. This shows that $\gamma$ was not maximal. ∎

## 21. The Lie derivative (Wed 10/16-Fri 10/18)

Recall that we have not yet attempted to answer the question: *what is the derivative of a vector field*? The question is tricky because it requires comparing tangent vectors at different (although nearby) points in the manifold. One way to do this is using the flow construction.

**Definition/Lemma 21.1.** *Given $X, Y \in \mathscr{X}(M)$, let $\theta_t$ be the flow of $X$. Define the* **Lie derivative** *of $Y$ with respect to $X$ by:*

$$(\mathscr{L}_X Y)_p = \lim_{t\to 0} \frac{Y_{\theta_t(p)} - (d\theta_t)_p Y_p}{t}, \tag{21.1}$$

*where the limit is taken with respect to the topology on $TM$. The limit exists for every $p \in M$ and defines a smooth vector field on $M$ equal to*

$$\left.\frac{d}{dt}\right|_{t=0} (\theta_{-t})_* Y. \tag{21.2}$$

*Proof.* Since $\theta$ is smooth in all variables, the expression (21.2) gives a smooth vector field on $M$. We need only show that (21.2) and (21.1) are equal. Using the formula (18.1) and the fact that $\theta_t = \theta_{-t}^{-1}$, we have

$$(21.2)_p = \lim_{t\to 0} \frac{(d\theta_{-t})_{\theta_t(p)} Y_{\theta_t(p)} - Y_p}{t}.$$

By continuity of $\theta_t$, we have

$$\lim_{t\to 0} (d\theta_t)_p = \mathrm{Id}.$$

We may insert this into the limit to obtain:

$$\begin{aligned}(21.2)_p &= \lim_{t\to 0} (d\theta_t)_p \frac{(d\theta_{-t})_{\theta_t(p)} Y_{\theta_t(p)} - Y_p}{t} \\ &= \lim_{t\to 0} \frac{(d\theta_t)_p (d\theta_{-t})_{\theta_t(p)} Y_{\theta_t(p)} - (d\theta_t)_p Y_p}{t} \\ &= \lim_{t\to 0} \frac{Y_{\theta_t(p)} - (d\theta_t)_p Y_p}{t} = (21.1)_p.\end{aligned}$$

∎

It turns out that the Lie derivative of a vector field is not a new object.

**Theorem 21.2.** $\mathscr{L}_X Y = [X, Y]$.

*Proof.* **Case 1.** Suppose $p$ is a regular point. By Lemma 20.4, we may choose coordinates such that
$$X = \frac{\partial}{\partial x^1}.$$
Then the flow of $X$ is given by
$$\theta_t(x^1, \ldots, x^n) = \left(x^1 + t, x^2, \ldots, x^n\right).$$
In particular, definition (21.1) just agrees with the usual definition of partial derivative in the $x^1$-direction, so we have
$$\mathscr{L}_X Y = \frac{\partial Y}{\partial x^1},$$
where the RHS is to be understood in local coordinates. On the other hand, by the local formula (16.4), where $X$ corresponds to the constant vector field $(1, 0, \ldots, 0)$, we also have
$$[X, Y] = \frac{\partial Y}{\partial x^1}.$$
So the two agree.
**Case 2.** $p \in \mathrm{Supp} X$. If $p$ belongs to the closure of the set of regular points of $X$, then the two sides are equal for points arbitrarily close to $p$. But each side is a continuous vector field, so the two must also agree at $p$.
**Case 3.** $p \notin \mathrm{Supp} X$. In this case, there exists a neighborhood $W \ni p$ such that $X|_W \equiv 0$. The flow of $X$ is identically constant on this neighborhood, so the Lie derivative is zero. Meanwhile the Lie bracket also clearly vanishes. □

**Corollary 21.3.** *(a)* $\mathscr{L}_X Y = -\mathscr{L}_Y X$

*(b)* $\mathscr{L}_X [Y, Z] = [\mathscr{L}_X Y, Z] + [Y, \mathscr{L}_X Z]$.

*(c)* $\mathscr{L}_{[X,Y]} Z = \mathscr{L}_X \mathscr{L}_Y Z - \mathscr{L}_Y \mathscr{L}_X Z$.

*(d) For $f, g \in C^\infty(M)$, we have*
$$\mathscr{L}_{fX}(gY) = fg\mathscr{L}_X Y + fX(g)Y - gY(f)X.$$

*(e) Suppose that $X_1, Y_1$ are $F$-related to $X_2, Y_2$, respectively. Then $\mathscr{L}_{X_1} Y_1$ is $F$-related to $\mathscr{L}_{X_2} Y_2$.*

*(f) If $F$ is a diffeomorphism then we have*
$$F_* \mathscr{L}_X Y = \mathscr{L}_{F_* X} F_* Y.$$

*Proof.* These all follow from the already-established properties of the Lie bracket. For example, (c) and (d) just rephrase the Jacobi identity as in (16.3).[8] □

[8] On the other hand, one can give a proof of the Jacobi identity directly from the definition (21.1), as you will do on homework. This gives an "explanation" for the Jacobi identity based on the fact that the Lie bracket is indeed a kind of derivative.

**Remark 21.4.** Despite the fact that we already knew the Lie bracket, the reinterpretation as Lie derivative is extremely useful. Let's also point out that the Lie derivative makes sense on functions and agrees with the derivative that we already know:

$$\begin{aligned}(\mathscr{L}_X f)_p &= \lim_{t\to 0} \frac{f(\theta_t(p)) - f(p)}{t} \\ &= \lim_{t\to 0} \frac{f(\gamma_p(t)) - f(p)}{t} \\ &= (f\circ \gamma_p)'(0) \\ &= X_p(f).\end{aligned}$$

So the Lie derivative on functions is just the ordinary derivative:

$$\mathscr{L}_X f = X(f).$$

Later we will define the Lie derivative on general tensors, in particular on differential forms where it is most useful.

## 22. Commuting vector fields (Fri 10/18-Mon 10/21)

We now continue to discuss the relationship between brackets and flows. The following Lemma shows that the definition (21.2) also works away from $t = 0$.

**Lemma 22.1.** *For all $t_0$ such that $(p,t) \in \mathscr{D}$, we have*

$$\left.\frac{d}{dt}\right|_{t=t_0} (\theta_{-t})_* Y = (\theta_{-t_0})_* \mathscr{L}_X Y.$$

*Proof.* Using the chain rule, we have

$$\begin{aligned}LHS &= \left.\frac{d}{dt}\right|_{t=t_0} (\theta_{-t_0}\circ\theta_{t-t_0})_* Y \\ &= (\theta_{-t_0})_* \left.\frac{d}{dt}\right|_{t=t_0} (\theta_{t-t_0})_* Y.\end{aligned}$$

Setting $s = t - t_0$, we can identify the inside limit as $\mathscr{L}_X Y$. □

**Proposition 22.2.** *TFAE:*

(a) *$X$ and $Y$ commute (i.e. $[X,Y] = 0$)*

(b) *$Y$ is invariant under the flow of $X$*

(c) *$X$ is invariant under the flow of $Y$.*

*Proof.* (b) ⇒ (a) is obvious from the definition (21.1), since the difference quotient vanishes identically. The same goes for $(c) \Rightarrow (a)$, since $[Y,X] = -[X,Y]$.

To show (a) ⇒ (b), assume that $\mathscr{L}_X Y = 0$. By the previous lemma, we have

$$\frac{d}{dt}(\theta_{-t})_* Y = (\theta_{-t})_* \mathscr{L}_X Y = 0.$$

So $(\theta_{-t})_* Y = Y$ for all $t$, i.e. $Y$ is invariant under the flow of $X$. (a) ⇒ $(c)$ is the same. □

**Corollary 22.3.** *Every vector field is invariant under its own flow.*

**Corollary 22.4.** *$X$ and $Y$ commute if and only if their respective flows $\theta_t$ and $\eta_s$ commute, assuming they are defined on the rectangle $[0,t]\times[0,s]$.*[9]

*Proof.* (⇒) It suffices to show that the curve

$$t \mapsto (\eta_s \circ \theta_t)(p)$$

is an integral curve of $X$. For, $t \mapsto (\theta_t \circ \eta_s)(p)$ is also an integral curve of $X$ by definition; since both begin at $\eta_s(p)$, they must be equal for all $t$, giving $\eta_s(\theta_t(p)) = \theta_t(\eta_s(p))$.

To this end, we calculate

$$\frac{d}{dt}(\eta_s \circ \theta_t)(p) = (\eta_s)_* \frac{d}{dt}(\theta_t(p)) = (\eta_s)_* X_{\theta_t(p)}$$

By the previous Lemma, since $X$ and $Y$ commute, $X$ is invariant under the flow of $Y$, i.e. $(\eta_s)_* X = X$. In particular, the RHS is equal to $X_{\eta_s(\theta_t(p))}$. This shows that $t \mapsto (\eta_s \circ \theta_t)(p)$ is an integral curve of $X$, so we are done.

(⇐) Assume that the flows commute, so $\theta_t \circ \eta_s = \eta_s \circ \theta_t$. We calculate

$$Y_{\theta_t(p)} = \frac{d}{ds}\bigg|_{s=0} \eta_s(\theta_t(p)) = \frac{d}{ds}\bigg|_{s=0} (\theta_t(\eta_s(p))) = (d\theta_t)_p Y_p.$$

By the definition (21.1), this implies $\mathscr{L}_X Y = 0$. □

We now come to the following important generalization of Lemma 20.4. This provides an answer to the question: *when is a collection of vector fields equal to the coordinate vector fields in some system of coordinates?* A necessary condition is that these vector fields all commute, since this is automatically true of coordinate vector fields. The theorem says that this is also sufficient.

**Theorem 22.5.** *Let $X_1, \dots, X_k$, $k \le n$, be a collection of commuting vector fields ($[X_i, X_j] = 0$ for all $i,j$) which are linearly independent at $p \in M$. There exists a local coordinate system $\{s^1, \dots, s^n\}$ near $p$ such that*

$$X_i = \frac{\partial}{\partial s^i}$$

*for $i = 1, \dots, k$. If $S$ is any codimension $k$ submanifold near $p$ for which $T_pS$ is complementary to $\langle (X_1)_p, \dots, (X_k)_p \rangle$, we may further choose the coordinates so that $S = \{s^i = 0, i = 1, \dots, k\}$.*

*Proof.* The proof is a simple generalization of the proof of Lemma 20.4. Let $U = \{x^1, \dots, x^n\}$ be a slice chart for $S$, in which $S = \{x^1 = \dots = x^k = 0\}$. (If no $S$ is given, choose $S$ to be a coordinate plane in an appropriate coordinate chart.) Let $\theta^i$ be the flow of $X_i$. Define a smooth map from a neighborhood $V$ of the origin in $\mathbb{R}^n$ to $U$ by

$$\Phi(s^1, \dots, s^n) = \theta^1_{s^1} \circ \theta^2_{s^2} \circ \cdots \circ \theta^k_{s^k}(0, \dots, 0, s^{k+1}, \dots, s^n) \in U.$$

[9]See Lee, p. 232-3 for a precise statement.

For $i \le k$, we have

$$\begin{aligned}\frac{\partial\Phi}{\partial s^i} &= \frac{\partial}{\partial s^i}\theta^1_{s^1}\circ\theta^2_{s^2}\circ\cdots\circ\theta^k_{s^k}(0,\dots,0,s^{k+1},\dots,s^n)\\ &= \frac{d}{ds^i}\left(\theta^i_{s^i}\circ\theta^1_{s^1}\circ\cdots\circ\hat{\theta}^i_{s_i}\circ\cdots\circ\theta^k_{s^k}\left(0,\dots,0,s^{k+1},\dots,s^n\right)\right),\end{aligned}$$

since the $\theta^i$'s commute. Since $\theta^i_{s_i}$ is the flow of $X_i$, this gives us

$$\begin{aligned}\frac{\partial\Phi}{\partial s^i} &= (X_i)_{\theta^i_{s^i}\circ\theta^1_{s^1}\circ\cdots\circ\hat{\theta}^i_{s_i}\circ\cdots\circ\theta^k_{s^k}(0,\dots,0,s^{k+1},\dots,s^n)}\\ &= (X_i)_{\Phi(s^1,\dots,s^n)}.\end{aligned}$$

This translates to

$$\Phi_*\frac{\partial}{\partial s^i} = X_i. \tag{22.1}$$

At the origin, for $j > k$, we have

$$\left.\frac{\partial\Phi}{\partial s^j}\right|_0 = \begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}$$

by construction, where the 1 is in the $j$'th row. In other words,

$$\Phi_*\left.\frac{\partial}{\partial s^j}\right|_p = \frac{\partial}{\partial x^j}.$$

Since $\{(X_1)_p,\dots,(X_k)_p,\frac{\partial}{\partial x^{k+1}},\dots,\frac{\partial}{\partial x^n}\}$ are linearly independent at the origin, the inverse function theorem tells us that $\Phi$ (or more accurately, $\Phi^{-1}$) is a coordinate system. Then (22.1) translates to

$$(\Phi^{-1})_*X_i = \frac{\partial}{\partial s^i}$$

for $i \le k$, as required. □

**Remark 22.6.** There is an important generalization called the *Frobenius theorem*, which provides an answer to the question: *when is a collection of vector fields $X_1,\dots,X_k$ locally tangent to a (family of) k-dimensional submanifold(s)*? We hope to discuss this later once we have more machinery available.

# Part 5. Lie groups and Lie algebras

## 23. Definition and examples (Mon 10/21)

**Definition 23.1.** A **topological group** is a topological space $G$ endowed with continuous maps

$$\begin{aligned} m : G \times G &\to G \\ (g,h) &\mapsto g \cdot h \end{aligned}$$

and

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

which give $G$ the structure of a group. $G$ is further called a **Lie group** if it is a smooth manifold and these maps are smooth.

**Examples 23.2.**

1. $(V,+)$, where $V$ is a finite-dimensional vector space with its canonical smooth structure (Example 3.5)).
2. $K^\times = K \setminus \{0\}$ under multiplication, where $K = \mathbb{R}$ or $\mathbb{C}$.
3. $S^1 \subset \mathbb{C}^\times$.
4. $T^n = S^1 \times \cdots \times S^1$.
5. $\mathrm{GL}(n,K)$ (see Example 3.6).
6. $\mathrm{GL}^+(n,K) \subset \mathrm{GL}(n,K)$, the set of matrices with positive determinant.
7. The group of invertible upper-triangular matrices.
8. $\mathrm{SL}(n,K) := \{A \in \mathrm{GL}(n,K) \mid \det(A) = 1\}$. This is clearly a subgroup; to prove that it is a Lie group, it is sufficient to prove that it is a smooth submanifold, i.e. that 1 is a regular value of the function $\det(\cdot) : \mathrm{GL}(n,K) \to K$.

   Let $A \in \mathrm{GL}(n,K)$. Denote by $m^i{}_j$ the determinant of the $i,j$'th minor of the matrix $A$, i.e. the $(n-1)\times(n-1)$ matrix obtained by deleting the $i$'th row and $j$'th column.

   Fixing $i$, we can calculate the determinant using expansion by minors:

   $$\det(A) = (-1)^{i+1} a^i{}_1 m^i{}_1 + (-1)^{i+2} a^i{}_1 m^i{}_2 + \cdots + (-1)^{i+n} a^i{}_n m^i{}_n.$$

   Notice that $m^i_k$ does not involve $a^i{}_j$ for any $k$. We therefore have

   $$\frac{\partial \det A}{\partial a^i{}_j} = (-1)^{i+j} m^i{}_j.$$

   In particular, $A$ is a critical point of $\det(A)$ (i.e. all partials vanish) if and only if all minors vanish. In this case the above formula implies that $\det(A) = 0$. But then $A \notin \mathrm{SL}(n,K)$. So $\det(\cdot)$ satisfies the Jacobian criterion on $\mathrm{SL}(n,K)$, which is therefore a smooth manifold of dimension $n^2 - 1$ $(K = \mathbb{R})$ or $2(n^2-1)$ $(K = \mathbb{C})$.

9. $\mathrm{O}(n) := \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A^T A = I_n\}$. This is the subgroup of orthogonal matrices. We must check that the defining equation $A^T A = I_n$ has full rank. In fact, it only has full rank considered as a map to the $n(n+1)/2$-dimensional subspace of symmetric matrices:

$$f : \mathrm{GL}(n,\mathbb{R}) \to S_n = \{A \in M_{\mathbb{R}}^{n\times n} \mid A^T = A\}$$
$$A \mapsto A^T A.$$

To show that $A$ is a regular point, we calculate

$$df_A(X) = X^T A + A^T X.$$

Given $B \in S_n$, we must exhibit $X$ such that $dF_A(B) = X$. In fact, it is sufficient to solve

$$A^T X = \frac{1}{2} B,$$

since then also

$$X^T A = (A^T X)^T = \frac{1}{2} B^T = \frac{1}{2} B.$$

Since $\det(A) = \pm 1$, we can take

$$X = \frac{1}{2}(A^T)^{-1} B.$$

We have shown that $\mathrm{O}(n)$ is a closed submanifold of dimension

$$n^2 - n(n+1)/2 = \frac{n(n-1)}{2}.$$

It is in fact compact, because $\operatorname{Tr} A^T A = n$ is the sum of the norms of the columns.

10. The same argument works over $\mathbb{C}$, replacing $A^T$ by $A^\dagger$ and $S_n$ by the space of Hermitian matrices. This gives the **Unitary Group** $\mathrm{U}(n)$.

11. Since $\det(A) = \pm 1$ on $\mathrm{O}(n)$, the set with $\det(A) = 1$ is an open submanifold / subgroup, called the **Special Orthogonal Group** $\mathrm{SO}(n)$.

12. Any countable group with the discrete topology.

## 24. Lie-group actions, homomorphisms, subgroups (Wed 10/23)

Today we shall upgrade several standard group-theoretic notions to the world of Lie groups. This is usually just a question of adding "smooth" somewhere in the definition, but the consequences of doing so can be nontrivial.

### 24.1. Lie-group actions.

**Definition 24.1.** Recall that a (left) **group action** by a group $G$ on a space $M$ is a map

$$G \times M \to M$$
$$(g, p) \mapsto g \cdot p$$

which satisfies

- $e \cdot p = p$
- $g \cdot (h \cdot p) = (gh) \cdot p$

for all $p \in M$. If $G$ acts on the left on $M$, we write $G \circlearrowleft M$.

Supposing that $G$ is a Lie group and $M$ is a smooth manifold, the action is further called a **Lie-group action** if the above map is smooth.

A *right* Lie group action is defined in a similar way, except that the above equations are replaced by

$$M \times G \to M$$
$$(p, g) \mapsto p \cdot g$$

and

- $p \cdot e = p$
- $(p \cdot g) \cdot h = p \cdot (gh)$.

By default, our actions will be left-actions. One can convert a right-action into a left-action by replacing the action of $g$ by that of $g^{-1}$.

Note that for each $g \in G$, the map $p \mapsto g \cdot p$ is a diffeomorphism of $M$, since its inverse is given by the action of $g^{-1}$.

The **orbit** of $p \in M$ is denoted $G \cdot p = \{g \cdot p \mid g \in G\}$.

The **isotropy group** or **stabilizer** of $p \in M$ is denoted $G_p = \{g \in G \mid g \cdot p = p\}$.

The action is said to be **transitive** if $G \cdot p = M$ for all (or any) $p \in M$. Observe that for a transitive group action and any $p, q \in M$, the stabilizers $G_p$ and $G_q$ are conjugate, and in particular are isomorphic as Lie groups.

The action is said to be **free** if $G_p = \{e\}$ for all $p \in M$.

**Examples 24.2.**

1. $\mathrm{GL}(n, K) \circlearrowleft K^n$ by definition. The orbits are $K^n \setminus \{0\}$ and $\{0\}$.
2. $\mathrm{O}(n) \circlearrowleft \mathbb{R}^n$. The orbits are the spheres $S^n_r, r > 0$ and the point $\{0\}$.
3. Let $X$ be a complete vector field on $M$. This generates an action of $\mathbb{R}$ on $M$ by:

$$(t, p) \mapsto \theta_t(p),$$

as explained above. The orbits are the images of integral curves in $M$.

4. Every Lie group acts on itself both by left and by right-multiplication. These actions are denoted with special symbols: given $g \in G$, we write

$$L_g : G \to G \qquad R_g : G \to G$$
$$h \mapsto g \cdot h \qquad h \mapsto h \cdot g.$$

It is clear that $L_{g_1} \circ L_{g_2} = L_{g_1 g_2}$, so this is a left-action, while $R_g$ is a right-action. Note that unless the group is abelian, these are genuinely different actions and cannot simply be interchanged by the standard conversion $g \mapsto g^{-1}$.

These actions are both free and transitive.

5. For an example of a different flavor, recall that the universal cover $\tilde{M} \to M$ is again a smooth manifold. By a theorem from Lee, $\pi_1(M, p)$ is countable, so is a Lie group with the discrete topology. This acts on $\tilde{M}$ by smooth deck transformations, giving a properly discontinuous Lie-group action; in fact, $M$ is just the quotient manifold $\tilde{M}/\pi_1(M, p)$.

24.2. **Lie-group homomorphisms.** Let $G$ and $H$ be Lie groups. A map $F : G \to H$ is called a **Lie-group homomorphism** if it is a smooth group homomorphism, i.e.

$$F(g_1 g_2) = F(g_1)F(g_2).$$

It is called an **isomorphism** if it is also a diffeomorphism (in which case the inverse is also a homomorphism). The **kernel** $\ker F = F^{-1}(e)$ is the inverse image of the identity element in $H$.

**Examples 24.3.**

1. $\exp : (\mathbb{R}, +) \to (\mathbb{R}^\times, \cdot)$ is a Lie-group isomorphism. $\exp : (\mathbb{C}, +) \to (\mathbb{C}^\times, \cdot)$ is a local diffeomorphism with kernel $2\pi i\mathbb{Z}$.

2. $\det : \mathrm{GL}(n, K) \to K^\times$ is a Lie group homomorphism. Its kernel is $\mathrm{SL}(n, K)$.

3. **Conjugation** by a fixed element $g \in G$ is an invertible Lie-group homomorphism from $G$ to itself (a.k.a an automorphism):

$$C_g = L_g \circ R_{g^{-1}} : G \to G$$
$$h \mapsto ghg^{-1}.$$

The map $G \times G \to G$ by $(g, h) \mapsto C_g(h)$ is also a left-action of $G$ on itself.

**Theorem 24.4.** *Every Lie-group homomorphism has constant rank.*

*Proof.* Let $F : G \to H$ be a Lie-group homomorphism and let $e \in G$, $\tilde{e} \in H$ be the respective identity elements. Let $g_0 \in G$. It suffices to show that the rank of $dF_{g_0}$ is equal to that of $dF_e$. Letting $L_{g_0}$ be left translation by $g_0$, we have for all $g \in G$ :

$$F(L_{g_0}(g)) = F(g_0 g) = F(g_0)F(g) = L_{F(g_0)}(F(g)).$$

In other words

$$F \circ L_{g_0} = L_{F(g_0)} \circ F.$$

Taking the derivative and applying the chain rule, we have

$$dF_{g_0} \circ (dL_{g_0})_e = (dL_{F(g_0)})_{\tilde{e}} \circ dF_e. \tag{24.1}$$

But $(dL_{g_0})_e$ and $\left(dL_{F(g_0)}\right)_{\tilde{e}}$ are both isomorphisms, so this shows that $dF_{g_0}$ and $dF_e$ have the same rank, as desired. ☐

**Corollary 24.5.** *A Lie-group homomorphism is surjective (resp. bijective) if and only if it is a submersion (resp. isomorphism).*

*Proof.* It follows by contradiction from the Baire category theorem (HW/Lee's book) that a surjective map of constant rank in fact must have full rank, i.e. must be a submersion. If it is also bijective then the dimensions must be equal, making it a diffeomorphism. ☐

24.3. **Lie subgroups.** A **Lie subgroup** of $G$ is a pair $(H, \varphi)$ where $H$ is a Lie group and $\varphi : H \to G$ is an injective immersion that is also a Lie-group homomorphism.

If $\varphi$ is an embedding then $H$ is called an **embedded Lie subgroup**,[10] and in this case we identify $H$ with its image $\varphi(H) \subset G$, which is a submanifold, and omit the inclusion map $\varphi$. Notice that $H$ is an embedded Lie subgroup if and only if it is a submanifold which is closed under the group operations of $G$. We shall be almost exclusively concerned with embedded Lie subgroups in this class.

**Examples 24.6.**

1. $\mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$ and $U(n) \subset \mathrm{GL}(n, \mathbb{C})$ are embedded Lie subgroups, as we showed last class.

2. $\mathrm{SO}(n) \subset \mathrm{O}(n)$ is the inverse image of $+1$ under det, which is an open submanifold, so an embedded Lie subgroup of index two.

3. Given any Lie group $G$, let $G_0$ be the connected component which contains the identity element $e$. By Proposition 2.3, this is both closed and open in $G$, so is an open submanifold. It is also closed under multiplication, because $G_0 \times G_0$ is connected and the image of the multiplication map
$$G_0 \times G_0 \to G_0 \subset G$$
must land in the connected component of the identity; similarly for the inversion map. So $G_0 \subset G$ is a properly embedded Lie subgroup (of the same dimension). In particular, one can show that $\mathrm{SO}(n)$ is the identity component of $\mathrm{O}(n)$, and $\mathrm{GL}^+(n, \mathbb{R})$ is the identity component of $\mathrm{GL}(n, \mathbb{R})$.

4. For an example of a non-embedded Lie subgroup, consider $H = \mathbb{R}$ and define the homomorphism
$$\begin{aligned} \varphi : \mathbb{R} &\to \mathbb{T}^2 \\ t &\mapsto (e^{2\pi i t}, e^{2\pi i \alpha t}) \end{aligned}$$

[10] An embedded Lie subgroup is more frequently called a **closed (Lie) subgroup** because of Theorem 24.9 below.

for an irrational number $\alpha$. The image is in fact dense in $\mathbb{T}^2$.

**Proposition 24.7.** *For any Lie-group homomorphism $F : G \to H$, the kernel $\ker F$ is a properly embedded (i.e. closed) normal Lie subgroup of $G$.*

*Proof.* By Theorem 24.4, $F$ has constant rank, so every fiber is properly embedded by Theorem 13.11. The kernel of any homomorphism is a normal subgroup, so we are done. ☐

**Example 24.8.**

- The proposition gives another proof that $\mathrm{SL}(n, K)$ is an embedded Lie subgroup of $\mathrm{GL}(n, K)$. Since det is surjective onto $K^\times$, it is a submersion by Corollary 24.5, so $\mathrm{SL}(n, K) \lhd \mathrm{GL}(n, K)$ has codimension one if $K = \mathbb{R}$ and two if $K = \mathbb{C}$.

- A unitary matrix $A$ satisfies $A^\dagger A = I_n$. Since $\det A^T = \det A$, we have $\det A^\dagger = \det \overline{A^T} = \overline{\det A^T} = \overline{\det A}$. Taking the determinant of each side of $A^\dagger A = I_n$, we have
$$|\det A|^2 = 1.$$
Hence, the determinant defines a Lie-group homomorphism
$$\det : \mathrm{U}(n) \to \mathrm{U}(1) \subset \mathbb{C}^\times,$$
which is surjective. Let
$$\mathrm{SU}(n) := \ker \det(\cdot) \lhd \mathrm{U}(n).$$
This is a (real) codimension one properly embedded Lie subgroup.

We now mention a result that allows us to dispense with saying "properly" every time we say "embedded Lie subgroup;" instead we shall usually say "closed Lie subgroup." (In fact, even saying "Lie" is not necessary: see Lee Theorem 20.12.) Since all of the Lie subgroups that we consider are properly embedded by definition, we will not prove the result, but it is good to know about.

**Theorem 24.9.** *Suppose $(H, \varphi)$ is a Lie subgroup of $G$. Then $\varphi(H)$ is closed in $G$ if and only if $\varphi$ is an embedding. In particular, every embedded Lie subgroup is properly embedded.*

*Proof.* See Lee, Theorem 7.21. The main idea is that because $G$ acts transitively on itself by diffeomorphisms, the whole picture can be reduced to the picture near the identity, where one can work in a fixed slice chart. ☐

## 25. The Equivariant Rank Theorem (Fri 10/25)

Today we will give another, more modern, method to construct the classical groups $\mathrm{O}(n), \mathrm{U}(n), \mathrm{SU}(n)$, etc. While the method is elegant enough to be worth doing for its own sake, it will also bring us closer to the right perspective on Lie groups.

**Definition 25.1.** Suppose that $G \circlearrowright_\theta M$ and $G \circlearrowright_\varphi N$ are two different Lie group actions which we call $\theta$ and $\varphi$, respectively, to keep them straight. This means that each $g \in G$ acts by diffeomorphisms $\theta_g : M \to M$ and $\varphi_g : N \to N$.

A smooth map $F : M \to N$ is said to be **equivariant** with respect to these two actions if

$$F(\theta_g(p)) = \varphi_g(F(p))$$

for all $g \in G$ and $p \in M$. In other words, $F \circ \theta_g = \varphi_g \circ F$, i.e., the diagram

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow{\scriptstyle \theta_g} & & \downarrow{\scriptstyle \varphi_g} \\ M & \xrightarrow{F} & N \end{array} \tag{25.1}$$

commutes. Sometimes one says that $F$ **intertwines** the two actions. If $\theta = \varphi$, one often says that $F$ **commutes** with the action.

**Examples 25.2.**

1. Let $M = N = G$ and take both $\theta$ and $\varphi$ to the action of $G$ on itself by left-multiplication. Fix $g_0 \in G$, and let $F = R_{g_0} : G \to G$. We claim that $F$ is equivariant:

$$R_{g_0}(g \cdot h) = (g \cdot h) \cdot g_0 = g \cdot (h \cdot g_0) = g \cdot R_{g_0}(h).$$

In other words, left and right-multiplication "commute" with each-other. (Funnily enough, this is another way of stating the *associative* property of group multiplication.)

2. Let $F : G \to H$ be a group homomorphism. Let $G$ act on itself by left-multiplication, and define an action $G \circlearrowright H$ by:

$$g \cdot h = F(g)h.$$

Let's check that this is a group action:

$$g_1 \cdot (g_2 \cdot h) = g_1 \cdot (F(g_2)h) = F(g_1)F(g_2)h = F(g_1 g_2)h = (g_1 g_2) \cdot h.$$

Then $F$ is an intertwiner because

$$F(g_1 g_2) = F(g_1)F(g_2) = g_1 \cdot F(g_2).$$

So we see that a homomorphism is an equivariant map for a certain choice of group action.

We have the following generalization of Theorem 24.4.

**Theorem 25.3** (Equivariant Rank Theorem)**.** *Let $G \circlearrowright_\theta M$ and $G \circlearrowright_\varphi N$ be Lie group actions, and suppose that $\theta$ is transitive. Any equivariant smooth map $F : M \to N$ has constant rank.*

*In particular, if $F$ is surjective then it is a submersion; if $F$ is injective then it is an immersion; if $F$ is bijective then it is a diffeomorphism.*

*Proof.* Let $p, q \in M$ be arbitrary. By transitivity, we can choose $g \in G$ such that $\theta_g(p) = q$. Applying the functor $d(\cdot)$, the diagram (25.1) goes over to

$$\begin{array}{ccc} T_pM & \xrightarrow{dF_p} & T_{F(p)}N \\ {\scriptstyle (d\theta_g)_p}\downarrow & & \downarrow{\scriptstyle (d\varphi_g)_{F(p)}} \\ T_qM & \xrightarrow{dF_q} & T_{F(q)}N. \end{array}$$

Since $(d\theta_g)_p$ and $(d\varphi_g)_{F(p)}$ are isomorphisms, the rank of $dF_p$ must equal that of $dF_q$.

The remaining statements are true of all constant-rank maps. □

As a first application, for $p \in M$, consider the **orbit map**

$$\theta^{(p)} : G \to M$$
$$g \mapsto g \cdot p.$$

This map is equivariant from $G$ with its left-multiplication action to $M$. Since left-multiplication is transitive, the previous theorem gives

**Corollary 25.4.**
- *For any $p \in M$, the stabilizer $G_p = (\theta^{(p)})^{-1}(p)$ is a closed Lie subgroup.*
- *Suppose $G_p = \{e\}$ for some $p \in M$. Then $\theta^{(p)} : G \to M$ is an injective immersion.*

Next, we will use the Equivariant Rank Theorem to construct our favorite Lie groups again while also calculating their dimensions in an easier way.

**Example 25.5.** Recall that $\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I_n\}$. Define

$$\Phi : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Mat}_{\mathbb{R}}^{n \times n}$$
$$A \mapsto A^T A.$$

Let $\mathrm{GL}(n, \mathbb{R})$ act on itself by *right*-multiplication, and act on $X \in \mathrm{Mat}_{\mathbb{R}}^{n\times n}$ by:

$$X \cdot B = B^T X B.$$

We have

$$\Phi(AB) = (AB)^T AB = B^T A^T AB = B^T \Phi(A) B = \Phi(A) \cdot B,$$

so $\Phi$ intertwines the two actions. By the equivariant rank theorem, $\mathrm{O}(n) = \Phi^{-1}(I_n)$ is a closed Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ (indeed it is closed in $\mathrm{Mat}_{\mathbb{R}}^{n\times n}$).

To determine the dimension of $\mathrm{O}(n)$, we need only calculate the tangent space at the identity element $e = I_n$. Since

$$d\Phi_{I_n}(X) = X^T I_n + I_n^T X = X^T + X,$$

we have

$$T_e\mathrm{O}(n) = \{X \in \mathrm{Mat}_{\mathbb{R}}^{n\times n} \mid X^T = -X\} =: o(n).$$

This is the subspace of **skew-symmetric matrices**, which has dimension

$$\dim o(n) = \dim \mathrm{O}(n) = \frac{n(n-1)}{2}.$$

**Example 25.6.** Generalizing the previous example, let $J$ be any $n \times n$ real matrix and define

$$\Phi(A) = A^T J A.$$

The last proof shows that $\{A \in \mathrm{GL}(n, K) \mid A^T J A = J\}$ is a Lie group. For instance, letting $J$ be a symmetric matrix of signature $(p,q)$, we obtain the group $\mathrm{O}(p,q)$ of linear transformations preserving a symmetric bilinear form of signature $(p,q)$. You will calculate its dimension on homework. A well-known example is the 6-dimensional *Lorentz group* $\mathrm{O}(1,3)$ which arose in the special theory of relativity.

**Example 25.7.** Recall that $\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A^\dagger A = I_n\}$. Define

$$\begin{aligned} \Phi : \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{Mat}_{\mathbb{C}}^{n\times n} \\ A &\mapsto A^\dagger A. \end{aligned}$$

Let $\mathrm{GL}(n,\mathbb{R})$ act on itself by *right*-multiplication, and act on $X \in \mathrm{Mat}_{\mathbb{R}}^{n\times n}$ by:

$$X \cdot B = B^\dagger X B.$$

As above, $\Phi$ is an intertwiner, so $\mathrm{U}(n)$ is a closed Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$ (indeed it is closed in $\mathrm{Mat}_{\mathbb{C}}^{n\times n}$). The tangent space at the identity is

$$T_e \mathrm{U}(n) = \{X \in \mathrm{Mat}_{\mathbb{C}}^{n\times n} \mid X^\dagger = -X\} =: u(n).$$

This is the subspace of **skew-Hermitian matrices**, of the form

$$\begin{pmatrix} ia_1 & & B \\ & \ddots & \\ B^\dagger & & ia_n \end{pmatrix},$$

where $a_1, \ldots, a_n$ are real numbers and $B$ contains arbitrary complex numbers. So the real dimension is

$$\dim u(n) = \dim \mathrm{U}(n) = n + 2\frac{n(n-1)}{2} = n^2.$$

**Example 25.8.** Last, we showed last time that $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det(A) = 1\}$ has codimension one in $\mathrm{U}(n)$, so

$$\dim \mathrm{SU}(n) = n^2 - 1.$$

Let us also compute the tangent space at the identity.

**Lemma 25.9.** $d(\det)_I(X) = \mathrm{Tr}\, X$.

*Proof.* There are several ways to prove this, the simplest being to observe that

$$\begin{aligned} \det(I + tX) &= \prod_{i=1}^n (1 + tX^i{}_i) + O(t^2) \\ &= 1 + t\sum_{i=1}^n X^i{}_i + O(t^2), \end{aligned}$$

since the sum defining the determinant can never have exactly one off-diagonal entry. □

For $K = \mathbb{C}$, det is a holomorphic function of the matrix entries, so the result still makes sense. We can conclude that

$$T_e\mathrm{SU}(n) = \{X \in \mathrm{Mat}_{\mathbb{C}}^{n\times n} \mid X^\dagger = -X, \mathrm{Tr}\, X = 0\} =: su(n)$$

is the space of **traceless skew-Hermitian matrices**, which of course has the same dimension

$$\dim su(n) = n^2 - 1.$$

**Remark 25.10.** We now make the following observation. Consider the case $u(n)$, and let $A, B$ be skew-Hermitian matrices. We have:

$$\begin{aligned}[A,B]^\dagger &= (AB)^\dagger - (BA)^\dagger \\ &= B^\dagger A^\dagger - A^\dagger B^\dagger \\ &= (-B)(-A) - (-A)(-B) \\ &= [B,A] \\ &= -[A,B].\end{aligned}$$

Hence, *the commutator of two skew-Hermitian matrices is again skew-Hermitian.* In other words, $u(n)$ is a *Lie algebra.* One can check that the same is true of $o(n)$, $o(p,q)$ (on homework), and $su(n)$. We will see next time why this is the case.

## 26. The Lie algebra of a Lie group (Mon 10/28-Wed 10/30)

26.1. **The definition.** A vector field $X$ on $G$ is said to be **left-invariant** if

$$(L_g)_* X = X$$

for all $g \in G$. The **Lie algebra** $\mathrm{Lie}(G)$ of $G$ is, by definition, the subspace of all left-invariant vector fields on $G$.[11]

It is easy to generate left-invariant vector fields on a Lie group: given a tangent vector at the identity $\xi \in T_eG$, we can define a (rough) vector field $X_\xi$ on $G$ by

$$(X_\xi)_g = (dL_g)_e(\xi)$$

for each $g \in G$. Let us check that $X_\xi$ is invariant under left-multiplication by $g_0 \in G$:

$$\begin{aligned}((L_{g_0})_* X_\xi)_{g_0 g} &= (dL_{g_0})_g (X_\xi)_g \\ &= (dL_{g_0})_g \left((dL_g)_e \xi\right) \\ &= (d(L_{g_0} \circ L_g))_e \xi \\ &= (dL_{g_0 g})_e \xi \\ &= (X_\xi)_{g_0 g} \checkmark.\end{aligned}$$

We can now characterize $\mathrm{Lie}(G)$ as follows.

[11]One could equally well define $\mathrm{Lie}(G)$ to be the set of right-invariant vector fields, but this would turn out to be slightly less convenient notationally.

**Lemma 26.1.** *(a) Any rough left-invariant vector field is smooth.*

*(b) The Lie bracket of two left-invariant vector fields is again left-invariant. In particular,* $\mathrm{Lie}(G) \subset \mathscr{X}(G)$ *is a Lie subalgebra.*

*(c) The map*

$$T_eG \to \mathrm{Lie}(G)$$
$$\xi \mapsto X_\xi$$

*is an isomorphism. In particular,* $\dim \mathrm{Lie}(G) = \dim G$.

*Proof.* (a) Let $X \in \mathrm{Lie}(G)$. By Proposition 16.3, it suffices to check that $X(f)$ is smooth for an arbitrary $f \in C^\infty(M)$. Choose a path $\gamma(t) \in G$ with $\gamma(0) = e$ and $\gamma'(0) = X_e$. We have

$$(Xf)(e) = \frac{d}{dt}\Big|_{t=0} f(c(t)).$$

Meanwhile, since $X$ is left-invariant, $t \mapsto g\gamma(t)$ is a path through $g$ whose derivative is $X_g$. So we also have

$$(Xf)(g) = \frac{d}{dt}\Big|_{t=0} f(g\gamma(t)).$$

Since $g\gamma(t)$ depends smoothly both on $g$ and $t$, $(Xf)(g)$ is a smooth function of $g$. This proves (a).

(b) This is clear from invariance of the Lie bracket under pushforward, Proposition 17.10.

(c) A left-inverse for the given map is:

$$\mathrm{Lie}(G) \to T_eG$$
$$X \mapsto X_e.$$

It is clear that the values of a left-invariant vector field at all points are determined by the value at a single point, so this is injective. □

In view of the Lemma, we are free to think of $\mathrm{Lie}(G)$ either as the subalgebra of $\mathscr{X}(G)$ consisting of global left-invariant vector fields, or simply as the tangent space at the identity $T_eG$ endowed with a canonical Lie-algebra structure. Later we will give the classical definition of the Lie bracket on $T_eG$, which has a more local flavor.

26.2. **The Lie algebra of** $\mathrm{GL}(n, K)$**.** Recall that $T_e\mathrm{GL}(n, K)$ is just the space $\mathrm{Mat}_K^{n\times n}$ of $n \times n$ matrices. We will now show that the Lie bracket on this space is just the ordinary commutator bracket of matrices. Moreover, for any subgroup $G < \mathrm{GL}(n, K)$, $T_eG \subset T_e\mathrm{GL}(n, K) = \mathrm{Mat}_K^{n\times n}$ is a subalgebra. This explains the observation of Remark 25.10.

**Proposition 26.2.** *The map* $\mathrm{Mat}_K^{n\times n} \to \mathrm{Lie}(\mathrm{GL}(n, K))$ *sending* $A$ *to* $X_A$ *is a Lie-algebra isomorphism.*[12]

[12] If you are not familiar with holomorphic tangent spaces, you can ignore the case $K = \mathbb{C}$ of this proposition. We will give an alternative proof next class using the exponential map.

*Proof.* A matrix $A = (A^i{}_j) \in \mathrm{Mat}_K^{n\times n}$ corresponds to the tangent vector

$$\sum_{i,j} A^i{}_j \frac{\partial}{\partial x^i{}_j}\bigg|_I \in T_I\mathrm{GL}(n,K).$$

To calculate the corresponding left-invariant vector field, let $\gamma(t) \in \mathrm{Mat}_K^{n\times n}$ be a path through $I$ with $\gamma'(0) = \sum_{i,j} A^i{}_j \frac{\partial}{\partial x^i{}_j}\Big|_I$. Let $g = (x^i{}_j) \in \mathrm{GL}(n,K)$. As in the proof of Lemma 26.1$a$, the path

$$(g\cdot\gamma(t))^i{}_j = \sum_k x^i{}_k(\gamma(t))^k{}_j$$

is tangent to the corresponding left-invariant vector field at $g$. We therefore have

$$(X_A)_g = \frac{d}{dt}\bigg|_{t=0} g\cdot\gamma(t) = \sum_{i,j,k} x^i{}_k A^k{}_j \, \frac{\partial}{\partial x^i{}_j}\bigg|_g . \tag{26.1}$$

This expression defines a smooth (indeed, linear) vector field on all of $\mathrm{Mat}_K^{n\times n}$, which is just Euclidean space with coordinates $x^i{}_j$, and we may compute the Lie bracket there. We have

$$[X_A,X_B] = \sum_{i_1,j_1,k_1,i_2,j_2,k_2} \left( x^{i_1}{}_{k_1} A^{k_1}{}_{j_1} \frac{\partial x^{i_2}{}_{k_2}}{\partial x^{i_1}{}_{j_1}} B^{k_2}{}_{j_2} - x^{i_1}{}_{k_1} B^{k_1}{}_{j_1} \frac{\partial x^{i_2}{}_{k_2}}{\partial x^{i_1}{}_{j_1}} A^{k_2}{}_{j_2} \right) \frac{\partial}{\partial x^{i_2}{}_{j_2}}.$$

Since $\frac{\partial x^{i_2}{}_{k_2}}{\partial x^{i_1}{}_{j_1}} = \delta^{i_2}{}_{i_1}\delta^{j_1}{}_{k_2}$, we may relabel $i = i_1 = i_2$ and $\ell = j_1 = k_2$, as well as $j = j_2$ and $k = k_1$. Factoring out $x^i{}_k$, we obtain

$$[X_A,X_B] = \sum_{i,j,k,\ell} x^i{}_k \left( A^k{}_\ell B^\ell{}_j - B^k{}_\ell A^\ell{}_j \right) \frac{\partial}{\partial x^i{}_j}.$$

Comparing with (26.1), we see that

$$[X_A,X_B] = X_{[A,B]},$$

which gives the result. □

The same proof works for any subgroup of $G < \mathrm{GL}(n,K)$, which can be seen as a submanifold of $\mathrm{Mat}_K^{n\times n}$ in its own right. We have also the following fact:

**Proposition 26.3.** *Supposing that $H < G$ is a Lie subgroup, the inclusion map $T_eH \subset T_eG$ is an injective Lie-algebra homomorphism, i.e., sends the Lie bracket on $T_eH \cong \mathrm{Lie}(H)$ to the Lie bracket on $T_eG \cong \mathrm{Lie}(G)$.*

*Proof.* We need only observe that given $\xi \in T_eH$, the corresponding left-invariant vector field $X_\xi^G$ on $G$ is tangent to $H$ (because it is the derivative of a path in $H$, see Proposition 13.13(1)) and left-invariant for the action of $H$, so restricts to $X_\xi^H$. Corollary 17.3 implies that the Lie brackets are the same. □

In view of these results, we will henceforth denote the space of $n \times n$ matrices over $K$, endowed with the commutator bracket, by

$$gl(n,K) = \mathrm{Mat}_K^{n\times n},$$

since it is canonically isomorphic to $\mathrm{Lie}(\mathrm{GL}(n,K))$. This also explains our above notations

$$o(n), u(n), su(n)$$

for the subalgebras of skew-symmetric, skew-hermitian, and traceless skew-hermitian matrices, respectively; these are canonically isomorphic to $\mathrm{Lie}(\mathrm{O}(n)), \mathrm{Lie}(\mathrm{U}(n))$, and $\mathrm{Lie}(\mathrm{SU}(n))$.

26.3. **Functoriality of** $\mathrm{Lie}(\cdot)$**.** We have the following generalization of Proposition 26.3

**Proposition 26.4.** *For a Lie-group homomorphism $F : H \to G$, the differential at the identity $dF_e : T_e H \to T_{\tilde{e}} G$ is a Lie-algebra homomorphism.*

*Proof.* We claim that for each $\xi \in T_e G$, $X^H_\xi$ is $F$-related to $X^G_{dF_e(\xi)}$. This is true by definition at the identity. Furthermore, recall from (24.1) that for $h \in H$, we have

$$dF_h \circ (dL_h)_e = (dL_{F(h)})_{\tilde{e}} \circ dF_e.$$

Applying this to $\xi$, we have

$$dF_h\left((dL_h)_e(\xi)\right) = (dL_{F(h)})_{\tilde{e}}(dF_e(\xi)).$$

But this just says that

$$dF_h(X^H_\xi)_h = (X^G_{dF_e(\xi)})_{F(h)}.$$

Since $h \in H$ was arbitrary, we are done.

The result then follows from the fact that the Lie bracket preserves $F$-relatedness, Proposition 17.10. □

This shows that taking tangent spaces/differentials at the identity is a functor from the category of Lie groups to the category of finite-dimensional real Lie algebras. It is very interesting to ask the converse question: is any finite-dimensional real Lie algebra the Lie algebra of a Lie group? Moreover, can one always lift a Lie-algebra homomorphism to a Lie-group homomorphism of the corresponding Lie groups?

The answer to the first question is in fact yes: there is a unique simply-connected Lie group with a given Lie algebra, up to canonical isomorphism. The answer to the second is in general no, but yes if the domain is simply connected. The moral is that up to passing to covers, the Lie algebra completely determines the Lie group.

We will try to prove these statements later if time permits.

## 27. Adjoint representation(s), exponential map (Wed 10/30-Fri 11/1)

27.1. **Left- versus right-multiplication.** Before proceeding, let's make a few more points about the Lie-algebra construction.

1. Let $X \in \mathrm{Lie}(G)$ be a left-invariant vector field and $g_0 \in G$. We of course have $(L_{g_0})_* X = X$. Meanwhile, $(R_{g_0^{-1}})_* X \in \mathrm{Lie}(G)$ is again left-invariant, but may not equal $X$:

$$\begin{aligned}(L_g)_*(R_{g_0^{-1}})_* X &= (L_g \circ R_{g_0^{-1}})_* X \\ &= (R_{g_0^{-1}})_*(L_g)_* X \\ &= (R_{g_0^{-1}})_* X.\end{aligned}$$

Hence, we obtain a left-action $G \circlearrowright \mathrm{Lie}(G)$ called the *Adjoint action* (see below). In fact, $(R_{g_0^{-1}})_*$ is a Lie-algebra homomorphism for each $g \in G$, since pushforward commutes with the Lie bracket. This is obviously something very special.

2. Let's also make note of a related fact / potential point of confusion. Let $g(t) \in G$ be a path with $g(0) = e$, and consider the family of diffeomorphisms $L_{g(t)}$. Since $L_{g(0)} = \mathrm{Id}$, the derivative
$$X = \frac{d}{dt} L_{g(t)}\Big|_{t=0} \in \mathscr{X}(M)$$
is a smooth vector field. One might think that this is left-invariant; however, it is actually *right*-invariant, because
$$\begin{aligned}\left((R_{g_0^{-1}})_* X\right)_h &= (R_{g_0^{-1}})_* \frac{d}{dt} L_{g(t)}(h)\Big|_{t=0} \\ &= \frac{d}{dt} R_{g_0^{-1}} L_{g(t)}(h)\Big|_{t=0} \\ &= \frac{d}{dt} L_{g(t)} R_{g_0^{-1}}(h)\Big|_{t=0} \\ &= X_{R_{g_0}^{-1}(h)}.\end{aligned}$$
On the other hand, the derivative of *right*-multiplication will be a *left*-invariant vector field; specifically, if $g'(t) = \xi$, we have $(e \cdot g)'(t) = g'(t) = \xi$, so we get
$$\frac{d}{dt} R_{g(t)}\Big|_{t=0} = X_\xi. \tag{27.1}$$
This shows that *any* left-invariant vector field is the derivative of right-multiplication by a path through the identity. This gives another useful way to describe $\mathrm{Lie}(G)$.

27.2. **The Adjoint/adjoint representations.** Recall that a **representation of a group**, $G$, over $K = \mathbb{R}$ or $\mathbb{C}$, is a group homomorphism $\rho : G \to \mathrm{GL}(V)$, where $V$ is a $K$-vector space. Equivalently, $\rho$ is a group action of $G$ on $V$ by $K$-linear maps, i.e., satisfies
$$\rho(gh) = \rho(g) \circ \rho(h).$$
This is called a **Lie-group representation** if $V$ is finite-dimensional and $\rho$ is smooth as a map from $G$ to $\mathrm{GL}(V)$.

A **representation of a Lie algebra**, $\mathfrak{g}$, is a Lie-algebra homomorphism $\rho : \mathfrak{g} \to gl(V)$, i.e., a map satisfying
$$\rho([\xi, \eta]) = [\rho(\xi), \rho(\eta)],$$
where the bracket on the left is the Lie bracket on $\mathfrak{g}$ and the bracket on the right is the commutator bracket on $gl(V)$.

Note that every Lie-group representation gives rise to a Lie-algebra representation by taking tangent spaces at the identity. This follows from functoriality of $\mathrm{Lie}(\cdot)$, Proposition 26.4.

We now attach a canonical pair of representations to any Lie group.

**Definition/Lemma 27.1.** *The (upper-case)* **Adjoint action** *of $G$ on* $\mathrm{Lie}(G)$*, denoted*

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$$

*is the representation defined in the following three equivalent ways:*

1. $\mathrm{Ad}_g = (R_{g^{-1}})_* : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$.
2. $\mathrm{Ad}_g = (dC_g)_e : T_eG \to T_eG$.
3. $\mathrm{Ad}_g : X_\xi \mapsto X_{(dL_g)_{g^{-1}}(dR_{g^{-1}})_e\xi}$.

*Proof.* We showed above that (1) is left-invariant, so we may indeed take the target to be $\mathrm{Lie}(G) \subset \mathscr{X}(M)$.

The fact that (2) and (3) are equivalent is obvious, because $C_g = L_g \circ R_g$.

It remains to show that (1) and (3) are equivalent. We have

$$\left(R_{g^{-1}}\right)_* X_\xi = (L_g)_*(R_{g^{-1}})_* X_\xi$$

by left-invariance. Taking the fiber at the identity gives the result. □

**Definition/Lemma 27.2.** *The (lower-case)* **adjoint representation** *of* $\mathrm{Lie}(G)$ *on itself, denoted*

$$\mathrm{ad} : \mathrm{Lie}(G) \to gl(\mathrm{Lie}(G))$$

*is the Lie-algebra representation defined by*

$$\mathrm{ad}_X(Y) = [X, Y].$$

*We have*

$$\mathrm{ad}_{(\cdot)} = (d\mathrm{Ad})_I(\cdot). \tag{27.2}$$

*Proof.* We first check that this is a Lie-algebra representation:

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$$

by Corollary 21.3$c$ (the Jacobi identity).

To check (27.2), let $X, Y \in \mathrm{Lie}(G)$ be left-invariant vector fields, and let $g(t)$ with $g(0) = e$ and $g'(0) = X_e$. By (a) of the previous theorem, we have

$$(d\mathrm{Ad}_e(X))(Y) = \frac{d}{dt}\Big|_{t=0} (R_{g^{-1}(t)})_* Y.$$

In view of (27.1), the family of diffeomorphisms $R_{g(t)}$ is equal to $\theta_t$, the flow of $X$, to second order at $t = 0$; it therefore computes the Lie derivative (exercise based on definition of the

Lie derivative). We also have $R_{g^{-1}(t)} = R_{g(-t)}$ to second order. So we get

$$\begin{aligned} d\mathrm{Ad}_e(X)(Y) &= \frac{d}{dt}\left(R_{g^{-1}(t)}\right)_* Y\Big|_{t=0} \\ &= \frac{d}{dt}\left(R_{g(-t)}\right)_* Y\Big|_{t=0} \\ &= \frac{d}{dt}(\theta_{-t})_* Y\Big|_{t=0} \\ &= [X, Y] = \mathrm{ad}_X(Y). \end{aligned}$$

□

**Remark 27.3.** Note that we can reverse the proof of the definition/lemma to give another derivation of the Jacobi identity!

27.3. **The exponential map.** We've shown how to go from a Lie group to a Lie algebra; let's briefly discuss how to go the other way. One can prove (HW) that any left-invariant vector field on a Lie group is complete, so it makes sense to define the **exponential map**

$$\begin{aligned} \exp : T_eG &\to G \\ \xi &\mapsto \gamma_e^{X_\xi}(1). \end{aligned}$$

Here $\gamma_e^{X_\xi}(t)$ is the integral curve of $X_\xi$ with $\gamma_e^{X_\xi}(0) = e$. The exponential map satisfies

$$\exp(t\xi) = \gamma_e^{X_\xi}(t),$$

which follows from the definition (HW), and

$$\exp((t+s)\xi) = \exp(t\xi) \cdot \exp(s\xi),$$

which follows from left-invariance. So for each $\xi \in T_eG$, one obtains a homomorphism $\mathbb{R} \to G$, a.k.a, a *1-parameter subgroup* of $G$. This is very useful, although it is important to note that in general

$$\exp(\xi + \eta) \neq \exp(\xi) \cdot \exp(\eta) \tag{27.3}$$

unless $\xi$ and $\eta$ commute.

In the case of a matrix Lie group $G < \mathrm{GL}(n, K)$, the exponential map is just the familiar one. Indeed, we saw in the proof of Proposition 26.2 that the left-invariant vector field $X_A$ corresponding to $A \in gl(n, K)$ is a linear vector field. As in Example 19.4, the integral curves are simply $t \mapsto \exp(tA)$, this being the matrix exponential. In fact, there is a precise formula expressing the relation between the two sides of (27.3), called the *Campbell-Baker-Hausdorff* formula, which turns out to involve only the commutator bracket. This corresponds to the fact that the Lie-algebra structure completely determines the Lie-group structure, locally, as we alluded to at the end of last class (and may get to prove later using the Frobenius theorem).

Now, we can identify all the tangent spaces of $\mathrm{GL}(n,K)$ with $gl(n,K)$ as usual. Then pushforward by left/right-multiplication just corresponds to matrix multiplication, so we have

$$\begin{aligned}\mathrm{Ad}_{\exp(tA)}(B) &= \exp(tA)B\exp(-tA)\\ &= (1+tA+\frac{t^2}{2}A^2+\cdots)B(1-tA+\frac{t^2}{2}A^2-\cdots)\\ &= 1+t(AB-BA)+\cdots.\end{aligned}$$

This shows us that the derivative of the Adjoint action at the identity indeed gives the adjoint representation:

$$\frac{d}{dt}\mathrm{Ad}_{\exp(tA)}(B)\Big|_{t=0} = \mathrm{ad}_A(B) = [A,B].$$

This is the classical viewpoint on the Lie bracket.

## 28. Example: SU(2) → SO(3) (Fri 11/1)

We shall now give a well-known example to demonstrate the above theory. You showed on homework that

$$\mathrm{SU}(2) = \left\{\begin{pmatrix} z & -\bar{w}\\ w & \bar{z}\end{pmatrix} \mid |z|^2+|w|^2=1\right\} < \mathrm{GL}(2,\mathbb{C}),$$

so that SU(2) is diffeomorphic to the 3-sphere. We also have

$$su(2) = \left\{\begin{pmatrix} ia & -\bar{w}\\ w & -ia\end{pmatrix} \mid a\in\mathbb{R}, w\in\mathbb{C}\right\} \subset gl(2,\mathbb{C}).$$

Let us label the four elements

$$\overbrace{q_0=\mathbf{1}=\begin{pmatrix}1 & \\ & 1\end{pmatrix},\ \underbrace{q_1=\begin{pmatrix}i & \\ & -i\end{pmatrix},\ q_2=\begin{pmatrix} & -1\\ 1 & \end{pmatrix},\ q_3=\begin{pmatrix} & i\\ i & \end{pmatrix}}_{su(2)}}^{\mathrm{SU}(2)}.$$

Conveniently, $q_1, q_2$, and $q_3$ belong both to SU(2) and to $su(2)$. Thinking of SU(2) as the 3-sphere, these three elements lie in the equatorial plane, which is also identified (by translation) with the tangent space at the north pole $q_0$. It is easy to check the following:

$$q_i^2 = -q_0,\ i=1,2,3, \qquad q_1q_2 = -q_3 \text{ \& cyclic permutations.}$$

These are called the **quaternion relations**, see more on homework.

Now, for $\omega,\eta\in gl(n,\mathbb{C})$, we can define

$$\langle\omega,\eta\rangle := \mathrm{Tr}\,\omega^\dagger\eta.$$

This is a Hermitian inner product:

$$\begin{aligned}\langle \omega, \eta \rangle &= \operatorname{Tr} \omega^\dagger \eta \\ &= \operatorname{Tr}(\omega^\dagger \eta)^T \\ &= \operatorname{Tr} \eta^T \bar{\omega} \\ &= \overline{\operatorname{Tr} \eta^\dagger \omega} = \overline{\langle \eta, \omega \rangle}.\end{aligned}$$

Specializing to $\omega, \eta \in su(n)$, we have

$$\begin{aligned}\langle \omega, \eta \rangle &= \operatorname{Tr} \omega^\dagger \eta \\ &= \operatorname{Tr}(-\omega)(-\eta^\dagger) \\ &= \operatorname{Tr} \eta^\dagger \omega \\ &= \langle \eta, \omega \rangle = \overline{\langle \omega, \eta \rangle}.\end{aligned}$$

So in fact $\langle \cdot, \cdot \rangle : su(n) \times su(n) \to \mathbb{R}$ is a real inner product. In particular, this gives us a natural inner product on the 3-dimensional real vector space $su(2)$, identifying it with Euclidean 3-space.

Now, given $A \in \mathrm{SU}(2)$ and $\omega, \eta \in su(2)$, we have

$$\begin{aligned}\langle \mathrm{Ad}_A \omega, \mathrm{Ad}_A \eta \rangle &= \operatorname{Tr}(A \omega A^\dagger)^\dagger (A \eta A^\dagger) \\ &= \operatorname{Tr} A \omega^\dagger A^\dagger A \eta A^\dagger \\ &= \operatorname{Tr} A \omega^\dagger \eta A^\dagger \\ &= \operatorname{Tr} A^\dagger A \omega^\dagger \eta \\ &= \operatorname{Tr} \omega^\dagger \eta = \langle \omega, \eta \rangle .\end{aligned}$$

So $\mathrm{Ad}_A$ acts by an isometry of $su(2) \cong \mathbb{R}^3$. The adjoint action therefore gives a group homomorphism

$$\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{SO}(3).$$

Note that these groups are both 3-dimensional; we claim that the map is a local diffeomorphism.

We first calculate the kernel. Note that

$$\mathrm{Ad}_{\pm q_0}(\omega) = (\pm q_0) \omega (\pm q_0) = q_0 \omega q_0 = \omega,$$

so $\{\pm q_0\} \subset \ker \mathrm{Ad}$; we claim that this is the entire kernel.

A general element of $\mathrm{SU}(2)$ conjugates $q_1$ as follows:

$$\begin{aligned}\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \begin{pmatrix} i & \\ & -i \end{pmatrix} \begin{pmatrix} \bar{z} & \bar{w} \\ -w & z \end{pmatrix} &= i \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \begin{pmatrix} \bar{z} & \bar{w} \\ w & -z \end{pmatrix} \\ &= i \begin{pmatrix} |z|^2 - |w|^2 & 2\bar{z}w \\ 2\bar{z}w & |w|^2 - |z|^2 \end{pmatrix}.\end{aligned}$$

If this equals $q_1$, we must have $z = 0$ or $w = 0$ based on the off-diagonal entries. If $z = 0$, then the result is $-q_1$, so we must have $w = 0$. Hence, the stabilizer of $q_1$ is the U(1) subgroup

$$\left\{ \begin{pmatrix} z & \\ & \bar{z} \end{pmatrix} \mid |z| = 1 \right\} < \mathrm{SU}(2).$$

We then calculate the action of this subgroup on $q_2$ :

$$\begin{aligned} \begin{pmatrix} z & \\ & \bar{z} \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} \bar{z} & \\ & z \end{pmatrix} &= \begin{pmatrix} z & \\ & \bar{z} \end{pmatrix} \begin{pmatrix} & -z \\ \bar{z} & \end{pmatrix} \\ &= \begin{pmatrix} & -z^2 \\ \bar{z}^2 & \end{pmatrix} \end{aligned}$$

This equals $q_2$ if and only if $z = \pm 1$, so we have $\ker \mathrm{Ad} = \{\pm q_0\}$ as claimed.

Since the kernel has dimension zero, we can already conclude that the (constant) rank is three, so Ad is a local diffeomorphism. It is 2-to-1 since the kernel has order two. Since any local isomorphism of (compact) groups is a covering map, we conclude that SU(2) is the universal cover of SO(3). Hence, we learn that

$$\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2.$$

Meanwhile, the derivative at the identity

$$\mathrm{ad} : su(2) \xrightarrow{\sim} so(3)$$

must be a Lie-algebra isomorphism! This gives us an example of two non-isomorphic Lie groups with isomorphic Lie algebras. Using the quaternion relations, it is easy to calculate this isomorphism explicitly in the basis $q_1, q_2, q_3$ for $su(2)$ :

$$\mathrm{ad}_{q_1}(\cdot) = [q_1, \cdot] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\mathrm{ad}_{q_2}(\cdot) = [q_2, \cdot] = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$\mathrm{ad}_{q_3}(\cdot) = [q_3, \cdot] = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Last, since $q_i^2 = -q_0$ for $i = 1, 2, 3$, the exponential map is given simply by

$$\exp(tq_i) = q_0 \cos t + q_i \sin t.$$

This is a 1-parameter subgroup traveling around a great circle in $\mathrm{SU}(2) \cong S^3$.

# Part 6. Vector bundles

## 29. Linear algebra (Mon 11/4)

29.1. **The Einstein summation convention.** Let $V$ be a finite-dimensional vector space over a field $K$. The *dual space* is denoted by $V^* = \mathrm{Hom}_K(V, K)$. We have a bilinear pairing

$$
\begin{aligned}
V \times V^* &\to K \\
(v, \alpha) &\mapsto \alpha(v).
\end{aligned}
\tag{29.1}
$$

This pairing gives rise to a canonical map from $V$ to $V^{**}$, also written

$$v \mapsto (\mathrm{ev}_v : \alpha \mapsto \alpha(v)).$$

Since $\dim(V) < \infty$, this is an isomorphism, so that $V$ and $V^{**}$ are canonically isomorphic. Meanwhile, $V$ and $V^*$ are isomorphic, but not in a canonical way.

There is, however, a canonical bijection between the set of *bases* for $V$ and the set of bases for $V^*$, as follows. Given a basis $\{e_i\}_{i=1}^n$ for $V$, there is a unique **dual basis** for $V^*$, denoted $\{e^j\}_{j=1}^n$, which satisfies

$$e^j(e_i) = \delta^j{}_i.$$

Let us see how the pairing (29.1) looks in these bases. Given $v = \sum_i a^i e_i \in V$ and $\alpha = \sum_j b_j e^j \in V^*$, we have

$$
\begin{aligned}
\alpha(v) = \sum_{i,j} a^i b_j e^j(e_i) &= \sum_{i,j} a^i b_j \delta^j{}_i \\
&= \sum_i a^i b_i.
\end{aligned}
$$

Let's see now how these look in a different basis, $\{\tilde{e}_i\}_{i=1}^n$, for $V$. Since this is a basis, there exists a matrix $\sigma = (\sigma^k{}_i)$ such that

$$e_i = \sum_k \sigma^k{}_i \tilde{e}_k.$$

We then have $e^j = \sum_\ell (\sigma^{-1})^j{}_\ell \tilde{e}^\ell$; for,

$$
\begin{aligned}
e^j(e_i) &= \sum_{k,\ell} (\sigma^{-1})^j{}_\ell \sigma^k{}_i \tilde{e}^\ell(\tilde{e}_k) \\
&= \sum_\ell (\sigma^{-1})^j{}_\ell \sigma^\ell{}_i = \delta^j{}_i.
\end{aligned}
$$

Now, we have

$$v = \sum a^i e_i = \sum a^i \sigma^k{}_i \tilde{e}_k = \sum_k \tilde{a}^k \tilde{e}_k,$$

where $\tilde{a}^k = \sum_i \sigma^k{}_i a^i$. We also have

$$\alpha = \sum_j b_j (\sigma^{-1})^j{}_\ell \tilde{e}^\ell = \sum \tilde{b}_\ell \tilde{e}^\ell,$$

where $\tilde{b}_\ell = \sum_j b_j (\sigma^{-1})^j{}_\ell$.

Let us now calculate

$$\begin{aligned}\sum_k \tilde{a}^k\tilde{b}_k &= \sum_{i,j,k} \sigma^k{}_i a^i b_j (\sigma^{-1})^j{}_k \\ &= \sum_{i,j,k} (\sigma^{-1})^j{}_k \sigma^k{}_i a^i b_j \\ &= \sum_{i,j} \delta^j{}_i a^i b_j \\ &= \sum_i a^i b_i.\end{aligned}$$

Is this a miracle? By no means: both sides are equal to $\alpha(v) \in K$, so they must agree.

In view of this everyday non-miracle we shall adopt the **Einstein summation convention** and simply write

$$\alpha(v) = a^i b_i,$$

where we omit the summation sign. The RHS refers to the sum of the coefficients of $v$ in the basis $\{e_i\}$ against the sum of the coefficients of $\alpha$ in, specifically, the *dual basis* $\{e^i\}$ of $\{e_i\}$. So the convention has a very precise meaning (that makes it work).

In addition to using the Einstein summation convention, we shall also *abuse* the Einstein summation convention, i.e., omit the summation sign when summing on upper and lower indices, even when these do not have precisely the meaning that we just explained. So e.g. the vector $v$ above will be written as

$$v = a^i e_i.$$

It's worth remembering that this is an abuse of the Einstein convention, not the real thing.

## 29.2. **Tensor products.**

29.2.1. *Motivation.* Let $V$ and $W$ be vector spaces over a field $K$. Given $\alpha \in V^*$ and $\beta \in W^*$, both linear maps to $K$, we can define a *bilinear* map

$$\begin{aligned} V \times W &\to K \\ (v, w) &\mapsto \alpha(v)\beta(w). \end{aligned} \tag{29.2}$$

This map is called "$\alpha \otimes \beta$;" we want to know what space it belongs to.

Since the sum of two bilinear functions is again bilinear (check), the set of all bilinear functions $V \times W \to K$ forms a vector space. We could simply define this space to be the "tensor product" of $V^*$ and $W^*$. However, the tensor product of $V$ with $W$ would then have to be defined by duality, which is a bit awkward. It's also not be entirely clear what sort of space we would be dealing with.

It is much better to give a construction of the tensor product that clearly satisfies a certain universal mapping property, as we shall do.

29.2.2. *Construction and universal property.* Let $S$ be a set. The **free vector space** on $S$ (over $K$) is the space of all formal $K$-linear combinations of elements of $S$:

$$\mathcal{F}(S) \coloneqq \{\sum_{i=1}^{m} a^i x_i \mid x_i \in S\}.$$

To be more precise, this is the space of all maps $f : S \to K$ with $f(x) = 0$ for all but finitely many $x \in S$, which is naturally a vector space (since $K$ is).

The **universal property of** $\mathcal{F}(S)$ is: given any map $A : S \to U$, where $U$ is a $K$-vector space, there exists a unique linear map $\bar{A} : \mathcal{F}(S) \to U$ such that the diagram

$$\begin{array}{ccc} S & & \\ \downarrow & \overset{A}{\searrow} & \\ \mathcal{F}(S) & \xrightarrow{\bar{A}} & U \end{array}$$

commutes. Here the vertical map sends $x \in S$ to the formal sum $1 \cdot x$. The universal property is quite obvious, e.g. just observe that the set of such $1 \cdot x$ is a basis for $\mathcal{F}(S)$, so can be mapped arbitrarily by a linear map.

Define $\mathcal{R} \subset \mathcal{F}(V \times W)$ to be the subspace generated by all elements of the forms

$$\begin{array}{ll} (av, w) - a(v, w) & \quad (v + v', w) - (v, w) - (v', w) \\ (v, aw) - a(v, w) & \quad (v, w + w') - (v, w) - (v, w'), \end{array}$$

where $a \in K, v \in V$, and $w \in W$. The **tensor product** of $V$ with $W$ is defined to be the quotient

$$V \otimes W = V \otimes_K W \coloneqq \mathcal{F}(V \times W)/\mathcal{R}.$$

The image of $(v, w) = 1 \cdot (v, w)$ is denoted by

$$v \otimes w = [(v, w)] \in V \otimes W.$$

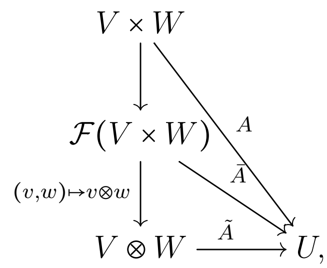
The **universal property of the tensor product** is as follows. Given a *bilinear* map

$$A : V \times W \to U,$$

where $U$ is a vector space, there exists a unique *linear* map $\tilde{A} : V \otimes W \to U$ such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & & \\ {\scriptstyle (v,w)\mapsto v\otimes w}\downarrow & \overset{A}{\searrow} & \\ V \otimes W & \xrightarrow{\bar{A}} & U. \end{array}$$

To prove the universal property of $\otimes$, first note that by the universal property of free vector spaces, there exists a unique map $\bar{A} : \mathcal{F}(V \times W) \to U$ such that the diagram above commutes. Then this map descends to $V \times W$ if and only if $\bar{A}$ vanishes on $\mathcal{R}$. But that is true if and only if $A$ itself is bilinear. To sum up, we get the following diagram:




as required. The map $\tilde{A}$ is unique because $\bar{A}$ is unique.

Note that the universal property characterizes $V \otimes W$ up to canonical isomorphism: given any other space satisfying the universal property, we can let $U = V \otimes W$ to get a unique (i.e. canonical) isomorphism.

29.2.3. *More properties of* $\otimes$.

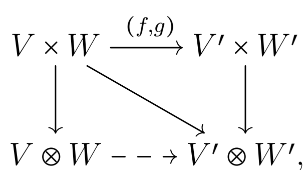
**Proposition 29.1** (Functoriality). *Given $f : V \to V'$ and $g : W \to W'$, linear maps, there exists a unique linear map*

$$f \otimes g : V \otimes W \to V' \otimes W'$$

*such that*

$$(f \otimes g)(v \otimes w) = f(v) \otimes g(w).$$

*Proof.* We have a diagram




where the diagonal map is just the composition. Notice that this is bilinear, because

$$\begin{aligned}(f(av + bv'), g(w)) &= (af(v) + bf(v')), g(w)) \\ &\mapsto (af(v) + bf(v')) \otimes w = af(v) \otimes w + bf(v') \otimes w,\end{aligned}$$

etc. So we can fill in the dotted arrow to obtain the desired map. Since the space of decomposable elements (i.e. those of the form $v \otimes w$) spans $V \otimes W$, the map is unique. □

**Proposition 29.2.** *Suppose $V$ and $W$ are finite-dimensional vector spaces over $K$ of dimension $m$ and $n$, respectively. Let $\{e_i\}, \{f_j\}$ be bases. Then $E = \{e_i \otimes f_j\}$ is a basis for $V \otimes W$. In particular,* $\dim(V \otimes W) = mn$.

*Proof.* To see that $E = \{e_i \otimes f_j\}$ spans the tensor product, recall by construction that decomposable elements span. But any decomposable element is clearly in the span of $E$, so $E$ in turn spans $V \otimes W$.

To check linear independence, suppose that

$$a^{ij} e_i \otimes f_j = 0;$$

we must show that $a^{ij} = 0$ for all $i, j$. Let $\{e^i\}$ and $\{f^j\}$ be the dual bases of $\{e_i\}$ and $\{f_j\}$, respectively. Consider the bilinear form $e^k \otimes f^\ell$ obtained as in (29.2), i.e., the one coming from the diagram

$$\begin{array}{ccc} V \times W & & \\ \downarrow & \searrow^{e^k(\cdot) f^\ell(\cdot)} & \\ V \otimes W & \xrightarrow{e^k \otimes f^\ell} & K. \end{array}$$

Applying this to the above sum, we get

$$0 = e^k \otimes f^\ell(a^{ij} e_i \otimes e_j) = a^{ij} e^k \otimes f^\ell(e_i \otimes f_j) = a^{k\ell}.$$

This completes the proof of linear independence. □

**Proposition 29.3.** *Assuming that $V$ and $W$ are finite-dimensional, we have canonical isomorphisms*

$$V^* \otimes W^* \cong (V \otimes W)^* \cong L(V, W; K),$$

*where $L(V, W; K)$ is the space of bilinear functionals $V \times W \to K$.*

*Proof.* Denote the spaces in the proposition by (1), (2), and (3).

To make a map (1) → (2), it suffices to make a bilinear map $V^* \times W^* \mapsto (V \otimes W)^*$. Given $(\alpha, \beta) \in V^* \otimes W^*$, we have as usual the bilinear map $\alpha \cdot \beta : V \times W \to K$, which descends to a linear map $\alpha \otimes \beta : V \otimes W \to K$. This map is bilinear in $\alpha$ and *beta*, so descends to the required map. A map (1) → (3) can be constructed in the same fashion.

To map (3) → (2), just observe that a bilinear map $V \times W \to K$ gives a linear map $V \otimes W \to K$. Of course, the composition of the maps (1) → (3) → (2) is equal to the map (1) → (2) that we constructed.

Finally, we must show that if $V$ and $W$ are finite-dimensional then the map (1) → (2) is an isomorphism. By the previous proposition, we can take a basis $\{e_i \otimes f_j\}$ for $V \otimes W$. Then the collection $\{e^i \otimes f^j\}$ is clearly the dual basis for $(V \otimes W)^*$. Since the elements $e^i \otimes f^j$ all lie in the image of (1) → (2), we have a surjection. Of course these are also a basis for $V^* \otimes W^*$, so we are done. □

**Proposition 29.4.** *We have the following associativity and distributivity properties:*

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3), \quad \textit{etc.}$$

$$(V_1 \otimes \cdots \otimes V_n)^* \cong V_1^* \otimes \cdots \otimes V_n^*.$$

*Proof.* The associativity must hold because both sides verify the required universal property. The second statement follows from the previous proposition by associativity and induction. □

In view of this proposition, we can unambiguously denote the multiple tensor product

$$V_1 \otimes \cdots \otimes V_k.$$

This space satisfies a universal mapping property with respect to *multilinear* maps $V_1 \times \cdots \times V_k \to U$ which is an obvious generalization of the above universal property.

29.2.4. *Hom and Tr via* $\otimes$.

**Proposition 29.5.** *There is a canonical isomorphism*

$$W \otimes V^* \cong \text{Hom}\,(V, W)$$

*as long as* $\dim(V), \dim(W) < \infty$.

*Proof.* We can map $(w, \alpha) \in W \times V^*$ to the rank-1 linear map $(v \mapsto \alpha(v)w)$. This is bilinear, so descends to a map between the given spaces. To see that it is an isomorphism, fix bases $\{e_i\}$ for $V$ and $\{f_j\}$ for $W$. Let $A \in W \otimes V^*$ and define the matrix

$$A^i{}_j = f^i(A(e_j)).$$

This is just the usual matrix corresponding of a linear map in fixed bases. We can then reconstruct $A$ by

$$A = A^i{}_j f_i \otimes e^j,$$

which is in the image of the LHS. □

**Remark 29.6.** As a comment, let's see how a linear map $A$ as in the previous proof acts on a vector $v = b^k e_k \in V$. We have

$$A(v) = a^i{}_j f_i \otimes e^j(b^k e_k) = A^i{}_j b^j f_i \in W.$$

The coefficient of $f_i$ is $A^i{}_j b^j$, which corresponds to the usual rule for multiplying a matrix by a vector. It also follows that the matrix of the composition $A \circ B$ is $A^i{}_k B^k{}_j$, giving the usual matrix multiplication rule.

**Proposition 29.7.**

- *There is a canonical map* $\text{Tr} : \text{End}(V) \to K$, *called the* **trace**, *which is given in any basis by* $\text{Tr}\, A = A^i{}_i$.
- *There is a canonical element* $\mathbf{1} \in V \otimes V^*$ *which is given in any basis by* $e_i \otimes e^i$.

*Proof.* For the first bullet, by the previous proposition, we have $\text{End}(V) \cong V \otimes V^*$. The map in question is induced by the duality pairing (29.1) via the universal property of the tensor product!

For the second bullet, this is just the image of the identity map $\text{Id} \in \text{End}(V)$. □

29.3. **Symmetric and alternating tensors.** Let $V^{\times k} = \overbrace{V \times \cdots \times V}^{k}$ and $V^{\otimes k} = \overbrace{V \otimes \cdots \otimes V}^{k}$. The transposition map

$$\sigma_{ij} : V^{\times k} \to V^{\times k}$$
$$(v_1, \ldots, v_i, \ldots, v_j, \ldots v_n) \mapsto (v_1, \ldots, v_j, \ldots, v_i, \ldots v_n)$$

descends to an automorphism of the tensor product

$$\sigma_{ij} : V^{\otimes k} \to V^{\otimes k}$$
$$v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n \mapsto v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

We denote the space of **symmetric elements**

$$\mathrm{Sym}^k V = \{\omega \in V^{\otimes k} \mid \sigma_{ij}(\omega) = \omega \ \forall i,j\} \subset V^{\otimes k}$$

and **alternating elements**

$$\Lambda^k V = \{\omega \in V^{\otimes k} \mid \sigma_{ij}(\omega) = -\omega \ \forall i,j\} \subset V^{\otimes k}.$$

For $k = 2$, we have $V^{\otimes 2} = \mathrm{Sym}^2 V \oplus \Lambda^2 V$, but in general this is false. We do have canonical projections onto each space, which are more natural to write down on the dual spaces:

$$\mathrm{Sym} : (V^*)^{\otimes k} \to \mathrm{Sym}^k V^*$$
$$(\mathrm{Sym}\,\omega)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \omega\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right).$$
$$\mathrm{Alt} : (V^*)^{\otimes k} \to \mathrm{Alt}^k V^*$$
$$(\mathrm{Alt}\,\omega)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\,\sigma\, \omega\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right).$$

Ordinarily a projection map requires a choice of complementary space, but these ones work as written (it is the transpose of the inclusion map $\Lambda^k V \hookrightarrow T^{\otimes k} V$).

We also have a symmetric product

$$\mathrm{Sym}^k V^* \otimes \mathrm{Sym}^\ell V^* \to \mathrm{Sym}^{k+\ell} V^*$$
$$(\alpha, \beta) \mapsto \alpha \cdot \beta := \mathrm{Sym}\,\alpha \otimes \beta$$

and, more importantly, the **wedge product**

$$\Lambda^k V^* \otimes \Lambda^\ell V^* \to \Lambda^{k+\ell} V^*$$
$$(\alpha, \beta) \mapsto \alpha \wedge \beta := \frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}\,\alpha \otimes \beta.$$

These are both associative, the first is commutative, and we have the well-known formula

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

The reason for the sign in the definition of $\wedge$ is so that if $\{e^i\}$ is the dual basis of $\{e_i\}$ then

$$\left(e^{i_1} \wedge \cdots \wedge e^{i_k}\right)(e_{i_1}, \dots, e_{i_i}) = 1$$

if $(i_1, \dots, i_k)$ are distinct. In other words, we have

$$e^{i_1} \wedge \cdots \wedge e^{i_k} = \sum_{\sigma \in S_n} \mathrm{sgn}\,\sigma\, e^{\sigma(i_1)} \otimes \cdots \otimes e^{\sigma(i_k)}.$$

**Proposition 29.8.** *The set of elements*

$$\{e^{i_1} \wedge \cdots \wedge e^{i_k} \mid i_1 < i_2 < \ldots < i_k\}$$

*forms a basis for* $\Lambda^k V^*$. *In particular,* $\dim \Lambda^k V^* = \binom{n}{k} = \dim \Lambda^k V$.

*Proof.* The elements clearly span the given space by multilinearity of the iterated wedge product, and linear independence can be seen by plugging in the $k$-tuple $(e_{i_1}, \dots, e_{i_k})$. ☐

29.4. **The determinant.** The symmetric and wedge products are functorial: a linear map $A : V \to V$ induces maps from $T^{\otimes k}V \to T^{\otimes k}V$ which commute with the transposition maps $\sigma_{ij}$, so induce maps from $\Lambda^k V \to \Lambda^k V$ compatible with compositions. Note that the top wedge power $\Lambda^n V$ is a one-dimensional $K$-vector space, and $A$ induces a map

$$\Lambda^n A : \Lambda^n V \to \Lambda^n V.$$

Any endomorphism of a one-dimensional vector space is a multiple of the identity. We define the **determinant** $\det(A) \in K$ to be the unique element such that

$$\Lambda^n A = \det(A) \cdot \mathrm{Id}.$$

Now, given two endomorphisms $A, B : V \to V$, by functoriality, we have

$$\Lambda^n (A \circ B) = \Lambda^n A \circ \Lambda^n B.$$

The LHS is $\det(A \circ B)\mathrm{Id}$ and the RHS is $\det(A) \cdot \det(B)\mathrm{Id}$. We obtain the familiar relation

$$\det(A \circ B) = \det(A) \det(B).$$

## 30. Vector bundles (Wed 11/06)

30.1. **The definition.** Let $E$ and $X$ be topological spaces and $\pi : E \to X$ a surjective continuous map. We use the following notation:

$$\Gamma(U, E) = \{\text{sections of } \pi \text{ over } U\},$$

sometimes called *local sections*, and

$$\Gamma(E) = \Gamma(X, E),$$

called *global sections*. These will always be assumed to be continuous. Refer to Definition 16.1 for the definition of a section.

**Definition 30.1.** The triple $(E, X, \pi)$, usually abbreviated $E$, is called a **vector bundle** of rank $r$ over $K = \mathbb{R}$ or $\mathbb{C}$ if:

1. Each fiber $E_x$ is endowed with the structure of a vector space of dimension $r$ over $K$, in which the addition and multiplication maps $+ : E_x \times E_x \to E_x$ and $\cdot : K \times E_x \to E_x$ are both continuous in the subspace topology on $E_x$.

2. For each $x_0 \in X$, there exists a neighborhood $U \ni x_0$ together with a set of sections $e_1, \ldots, e_r \in \Gamma(U, E)$ such that $\{e_\alpha(x)\}$ forms a basis of $E_x$ for each $x \in U$, and the map

$$\begin{aligned} U \times K^r &\to \pi^{-1}(U) \subset E \\ (x, (a^1, \ldots, a^r)) &\mapsto a^\alpha e_\alpha(x) \end{aligned} \tag{30.1}$$

   is a homeomorphism.

With this definition comes the following terminology.

- $E$ is called the *total space*, $X$ is called the *base space*
- A pair $(U, \{e_\alpha\})$ as in (2) is called a *local frame* or *local trivialization*
- A vector bundle of rank one is called a *line bundle*. Care should be taken as to whether this is a real or complex line bundle, as the two have very different properties.
- If $E$ is a $\mathbb{C}$-vector bundle of rank $r$, we obtain a real vector bundle of rank $2r$ by letting $\mathbb{R} \subset \mathbb{C}$ act by restriction. This is called the *underlying real bundle*. For example, the underlying real bundle of a complex line bundle is a real vector bundle of rank two.

The reason for emphasizing the local frame $\{e_\alpha\}_{\alpha=1}^r$ in the previous definition is that a local frame is the analogue of a *basis* in the world of vector bundles, as shown by the following basic lemma.

**Lemma 30.2.** *Let $s \in \Gamma(V, E)$ be a section of $E$ over $V$. Given any local frame $(U, \{e_\alpha\})$, there exist continuous functions $s^\alpha \in C^0(U \cap V, E)$, $\alpha = 1, \ldots, r$, such that $s|_{U\cap V} = s^\alpha e_\alpha$.*

*Proof.* Since $e_\alpha(x)$ is a basis for $E_x$ for each $x \in X$, the functions $s^\alpha$ exist and are unique. It remains to check that they are continuous. This is true because each $s^\alpha$ is equal to the composition

$$s^\alpha : U \cap V \xrightarrow{s} \pi^{-1}(U \cap V) \xrightarrow{\sim} (U \cap V) \times K^r \xrightarrow{\pi_\alpha} K.$$

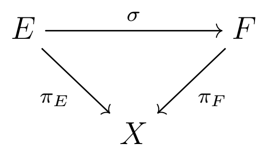
Here, the middle map is the inverse of the homeomorphism (30.1). □

## 30.2. **Examples.**

- $\underline{K}^r = X \times K^r$ is called the *product bundle* of rank $r$.
- $S = [0,1] \times \mathbb{R}/(0,v) \sim (1,-v)$ is called the **Möbius bundle**. It is diffeomorphic to the Möbius strip considered above, but we consider it as a bundle over $S^1 = [0,1]/0 \sim 1$ via the projection to the first factor.
- Take $X = K\mathbb{P}^n = \{\ell \subset K^{n+1} \mid \dim \ell = 1\}$. Define the **tautological bundle**
$$\mathscr{O}(-1) = \mathscr{O}_{K\mathbb{P}^n}(-1) := \{(\ell, v) \in X \times K^{n+1} \mid v \in \ell\}.$$
The projection map is the restriction of the projection to the first factor. For example, the tautological bundle over $\mathbb{RP}^1$ is the Möbius bundle (exercise on HW). For $K = \mathbb{C}$, the tautological bundle on $\mathbb{CP}^1$ is something (almost) new.
- If $M$ is a smooth manifold, the tangent bundle $TM$ is a vector bundle of rank $r$ over $M$. By construction, each coordinate chart of $M$ gives a local frame $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$. We typically reserve Latin indices $(i, j, k, \ldots)$ for coordinate frames on $TM$ or $T^*M$, and use Greek indices $(\alpha, \beta, \gamma, \ldots)$ for general frames on a general vector bundle.

**Remark 30.3.** Notice that $TM \to M$ is in fact a *smooth* vector bundle, i.e., all the objects appearing in the definition are smooth; see Definition 35.1 below.

30.3. **Bundle morphisms and triviality.** Given two bundles $E$ and $F$ over the same space, $X$, a **bundle morphism** is a continuous map $\sigma$ such that the diagram




commutes and the induced map $\sigma_x : E_x \to F_x$ is linear for each $x \in X$.

The morphism $\sigma$ is called an **isomorphism** if it has an inverse morphism.

**Example 30.4.** By definition, the inclusion map $\mathscr{O}(-1) \to \underline{K}^{n+1}_{K\mathbb{P}^n}$ is an injective morphism.

It is easy (and important) to characterize bundle isomorphisms.

**Lemma 30.5.** *A bundle morphism $\sigma : E \to F$ is an isomorphism if and only if $\sigma_x : E_x \to F_x$ is an isomorphism for each $x \in X$.*

*Proof.* ($\Rightarrow$) This direction is trivial.

($\Leftarrow$) If $\sigma_x$ is invertible for each $x$ then a set-theoretic inverse $\sigma^{-1}$ clearly exists; it remains to check continuity, which can be done in any local trivializations. Write $\sigma(e_\alpha) = \sigma^\beta{}_\alpha(x) f_\beta$, and $\sigma^{-1}(f_\beta) = \tau^\alpha{}_\beta e_\alpha$. We must show that $\tau^\alpha{}_\beta$ is continuous. We have

$$\begin{aligned} f_\beta(x) &= \sigma_x(\sigma_x^{-1}(f_\beta(x))) \\ &= \tau^\alpha{}_\beta(x)\sigma_x(e_\alpha(x)) \\ &= \tau^\alpha{}_\beta(x)\sigma^\gamma{}_\alpha(x) f_\gamma(x) \\ &= \sigma^\gamma{}_\alpha(x)\tau^\alpha{}_\beta(x) f_\gamma(x). \end{aligned}$$

This is true for each $x$ and $\beta = 1, \ldots, r$ if and only if

$$(\tau^\alpha{}_\beta)(x) = (\sigma^\alpha{}_\beta(x))^{-1}.$$

As we know from homework, the inverse of a continuous, invertible matrix-valued function is continuous (by Cramer's formula). ☐

**Definition/Lemma 30.6.** *A vector bundle $E \to X$ is called* **trivial** *if it is isomorphic to the product (a.k.a. trivial) bundle $\underline{K}^r = X \times K^r$. This is true if and only if $E$ admits a global frame.*

*Proof.* ($\Rightarrow$) The trivial bundle has the global frame $\{e_\alpha(x) = (x, (0, \ldots, 1, \ldots, 0))\}$, where the 1 is in the $\alpha$'th entry.

($\Leftarrow$) Suppose given a global frame $\{e_\alpha\}_{\alpha=1}^r \subset \Gamma(E)$. Define a morphism $\underline{K}^r \to E$ by sending

$$(x, (a^1, \ldots, a^r)) \mapsto a^\alpha e_\alpha(x).$$

Since $\{e_\alpha\}$ is a global frame, this is an isomorphism on each fiber. By the Lemma, it is a global isomorphism. ☐

**Definition 30.7.** A smooth manifold $M$ is called **parallelizable** if $TM$ is a trivial vector bundle, i.e., admits a global frame. This is a set of global vector fields $X_1, \ldots, X_n$ such that $(X_1)_p, \ldots, (X_n)_p$ spans $T_pM$ for each $p \in M$.

**Examples 30.8.**

- You showed on homework that $TS^1$ is diffeomorphic to $S^1 \times \mathbb{R}$. In the process you probably constructed the global frame $\frac{\partial}{\partial\theta}$ for $TS^1$, showing that it is parallelizable.
- $TS^2$ does not admit any nonvanishing global sections, by the hairy ball theorem. So $S^2$ is not parallelizable.
- $S^3 \cong \mathrm{SU}(2)$ is parallelizable, by the following proposition.

**Proposition 30.9.** *Every Lie group is parallelizable.*

*Proof.* A basis for $\mathrm{Lie}(G)$ forms a global frame. □

**Remark 30.10.** The $n$-sphere is parallelizable if and only if $n = 1, 3$, or $7$. This was proved independently by Hirzebruch, Kervaire, Bott, and Milnor in 1958.

## 31. Transition functions (Fri 11/08)

As with smooth manifolds, there are two perspectives on vector bundles: one abstract and one coordinate-based. The vector-bundle analogue of a coordinate chart is a local frame, and the analogue of a transition map is the following.

**Definition 31.1.** Let $E \to X$ be a vector bundle. Fix a cover $\{U_a\}_{a\in I}$ of $X$ such that $E|_{U_a}$ is trivial for each $a$, and pick a local frame $\{e^a_\alpha\}_{\alpha=1}^r \subset \Gamma(U_a, E)$ for each $a$. The **transition function** $\sigma_{ab} \in C^0\left(U_a \cap U_b, \mathrm{GL}\left(n, K\right)\right)$ is defined by:

$$e^a_\alpha = \sum_\beta {\sigma_{ab}}^\beta{}_\alpha e^b_\beta \quad \text{(no sum on } b).$$

Given a section $s \in \Gamma(U_a \cap U_b, E)$, let $s^\alpha$ and $t^\alpha$ be its local components in the frames $\{e^a_\alpha\}$ and $\{e^b_\beta\}$, respectively, so that

$$s^\alpha e^a_\alpha = s = t^\beta e^b_\beta.$$

We then have $s = s^\alpha {\sigma_{ab}}^\beta{}_\alpha e^b_\beta$ (no sum on $b$), so that

$${\sigma_{ab}}^\beta{}_\alpha s^\alpha = t^\beta.$$

In other words, the transition function just acts on components by matrix multiplication.

Note that by definition, we always have

$$\sigma_{aa} = \mathbf{1}$$

and, by composition,

$$\sigma_{bc} \cdot \sigma_{ab} = \sigma_{ac} \text{ on } U_a \cap U_b \cap U_c.$$

These are called **cocycle conditions**. With $a = c$, the second one also gives

$$\sigma_{ca} = \sigma_{ac}^{-1} \text{ on } U_a \cap U_c.$$

**Examples 31.2.** 1. Let $E = TM$. Suppose given two charts $\{x^i\}$ and $\{y^j\}$, and consider the two coordinate frames $\{\frac{\partial}{\partial x^i}\}$ and $\{\frac{\partial}{\partial y^j}\}$. By definition, the transition function is the unique matrix-valued function satisfying

$$\frac{\partial}{\partial x^i} = {\sigma_{xy}}^j{}_i \frac{\partial}{\partial y^j}.$$

We therefore have

$${\sigma_{xy}}^j{}_i = \frac{\partial y^j}{\partial x^i}.$$

In other words, *the transition functions of $TM$ are the Jacobians of the transition maps of $M$.*

2. Consider the Möbius bundle $S \to S^1$. Let $U_0 = (0,1)$ and $U_1 = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] / \sim$ . We can make sections

$$e(x) = 1$$

over $U_0$ and

$$f(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} < x \leq 1 \end{cases}$$

over $U_1$. The transition function is

$$\sigma_{01} = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ -1 & \frac{1}{2} < x < 1. \end{cases}$$

3. Consider the tautological bundle $\mathscr{O}(-1) \to \mathbb{CP}^1$. Let $U_1 = \{[z,1] \mid z \in \mathbb{C}\}$ and $U_0 = \{[1,w] \mid w \in \mathbb{C}\}$. We can make sections

$$e([z,1]) = (z,1)$$

and

$$f([1,w]) = (1,w).$$

Since $w = \frac{1}{z}$, the transition function is

$$\sigma_{01} = z^{-1}.$$

This explains the notation $\mathscr{O}(-1)$.

As with smooth manifolds (cf. Lemma 4.1), one can completely reconstruct a bundle from its transition functions; in fact, one can also construct new bundles, as we shall do.

**Lemma 31.3** (Vector-bundle construction lemma)**.** *Given an open cover $\{U_a\}_{a\in\mathcal{I}}$ and a collection of matrix-valued functions $\{\sigma_{ab} \in C^0(U_a \cap U_b, \mathrm{GL}(r,K))\}_{a,b\in\mathcal{I}}$ satisfying the cocycle conditions, there exists a vector bundle $E \to X$ with these transition functions.*

*Proof sketch.* We can simply let $E = \sqcup_a (U_a \times K^r) / \sim$, where

$$(x,v)_a \sim (x, \sigma_{ab}(v))_b.$$

The cocycle conditions imply that this is an equivalence relation, so the space is well defined and picks one point in each chart over $x \in X$. The projection to $X$ is well defined and its fibers are vector spaces of dimension $r$. Also, the frames on each trivialization descend to frames on $E$. □

We can also reduce the question of triviality/isomorphism to one of transition functions.

**Proposition 31.4.** *Let $E \to X$ and $F \to X$ be vector bundles of rank $r$ over the same space. Then $E$ and $F$ are isomorphic if and only if, after passing to a common refinement $\{U_a\}_{a\in\mathcal{I}}$, their respective transition functions $\sigma_{ab}$ and $\sigma'_{ab}$ are related by*

$$\sigma_{ab} = \tau_b^{-1}\sigma'_{ab}\tau_a$$

*for a collection of matrix-valued functions $\{\tau_a \in C^0(U_a, \mathrm{GL}(r,K))\}_{a\in\mathcal{I}}$.*

*Proof.* This is an exercise in the definitions (on HW). □

**Remark 31.5.** In the case $r = 1$, since $\mathrm{GL}(1,K) = K^\times$ is abelian, this actually shows that the set of isomorphism classes of line bundles on $X$ is equal to the sheaf cohomology group $H^1(X, \mathscr{C}^0(K^\times))$.

## 32. Bundle operations, subbundles (Fri 11/08-Mon 11/11)

32.1. **Bundle operations.** We now come to the following **Meta-Theorem:** any[13] functorial operation on the category of vector spaces gives rise to a functorial operation on the category of vector bundles.

**Example 32.1.** The direct sum of two bundles has total space equal to the fiber product

$$E \oplus F = E \times_X F = \{(v,w) \in E \times F \mid \pi_E(v) = \pi_F(w)\}.$$

The fiber is $E_x \oplus F_x$, and the trivializations are the obvious ones.

**Example 32.2.** Define the **tensor product** $E \otimes F$ to be the bundle of rank $rs$ with transition functions $\sigma_{ab} \otimes \mu_{ab}$, where $\sigma$ and $\mu$ are transition functions for $E$ and $F$, respectively, on a common refinement. To be sure that Lemma 31.3 permits this, we need to check the cocycle conditions. Applying functoriality of tensor products, Proposition 29.1, we get

$$(\sigma_{bc} \otimes \mu_{bc}) \circ (\sigma_{ab} \otimes \mu_{ab}) = (\sigma_{bc} \circ \sigma_{ab}) \otimes (\mu_{bc} \circ \mu_{ab}) = \sigma_{ac} \otimes \mu_{ac}.$$

So the tensor-product bundle exists. For each $x \in X$, the fiber $(E \otimes F)_x$ is naturally identified with $E_x \otimes F_x$. It is not hard to convince oneself that $E \otimes F$ is independent of the system of trivializations used to construct it; one can also prove this using a universal property (HW).

Here is a table with all the operations we discussed, together with the corresponding transition functions:

[13] To be taken with a grain of salt.

| Operation | Fiber | Rank | Transition function |
|---|---|---|---|
| $E \oplus F$ | $E_x \oplus F_x$ | $r+s$ | $\sigma \oplus \mu$ |
| $E \otimes_K F$ | $E_x \otimes F_x$ | $r \cdot s$ | $\sigma \otimes_K \mu$ |
| $E^*$ | $E_x^*$ | $r$ | $(\sigma^T)^{-1}$ |
| $\operatorname{Hom}_K(E,F)$ | $F_x \otimes E_x^*$ | $r \cdot s$ | $\mu \otimes (\sigma^T)^{-1}$ |
| $\Lambda^k E$ | $\Lambda^k E_x$ | $\binom{r}{k}$ | determinants of $k \times k$ minors of $\sigma$ |
| $\det E := \Lambda^r E$ | $\Lambda^r E_x$ | 1 | $\det \sigma$ |
| $\bar{E}$ | $\bar{E}_x$ | $r$ | $\bar{\sigma}$. |

*Note:* For a complex bundle $E$, $\bar{E}$ is the same underlying real bundle but with a new complex scalar multiplication defined by

$$\lambda \cdot_{\bar{E}} v := \bar{\lambda} \cdot_E v.$$

**Definition 32.3.** We make the following definitions:

$$\mathscr{O}(1) := \mathscr{O}(-1)^*$$

and

$$\mathscr{O}(n) := \begin{cases} \mathscr{O}(1)^{\otimes n} & n \in \mathbb{N} \\ \underline{K} & n = 0 \\ \mathscr{O}(-1)^{\otimes |n|} & n \in -\mathbb{N}. \end{cases}$$

In the case of $\mathbb{CP}^1$, by Example 31.2.3, notice that the transition function of $\mathscr{O}(1)$ between the stereographic charts is $(z^{-1})^{-1} = z$. In the case of line bundles, transition functions just multiply under tensor product. So the transition function of $\mathscr{O}(n)$ is $z^n$.

Let's discuss two of these operations a bit more. Given a frame $\{e_\alpha\}$ for $E$ over $U$, there exists a dual frame $\{e^\beta\}$ for $E^*$ over $U$ satisfying

$$e^\beta(e_\alpha)(x) = \delta^\beta{}_\alpha$$

for all $x \in U$. Notice that a section $\alpha \in \Gamma(U, E^*)$ is equivalent to a morphism to the trivial bundle:

$$\begin{aligned} E|_U &\to \underline{K}|_U \\ (x,v) &\mapsto (x, \alpha(v)). \end{aligned}$$

More generally, let $\operatorname{Hom}(E,F)$ denote the space of bundle morphisms $E \to F$. We have:

**Proposition 32.4.** $\operatorname{Hom}(E,F) \cong \Gamma(F \otimes E^*)$.

*Proof.* Given $\rho \in \operatorname{Hom}(E,F)$, we get an element $\rho_x \in \operatorname{Hom}(E_x, F_x) \cong F_x \otimes E_x^*$ for each $x \in X$, so a rough section of $F \otimes E^*$. We can check continuity in any frames, as usual. Let $\{e_\alpha\}$ and $\{f_\alpha\}$ be local frames for $E$ and $F$, respectively, and denote their dual frames by $\{e^\alpha\}$ and $\{f^\alpha\}$, respectively. Let

$$\rho^\beta{}_\alpha = f^\beta(\rho(e_\alpha)),$$

which is a continuous matrix-valued function. Then the section corresponding to $\rho$ is

$$\rho^\beta{}_\alpha f_\beta \otimes e^\alpha,$$

which is continuous. ∎

Another bundle operation to mention is **restriction**: given a bundle $\pi : E \to X$ and any subspace $S \subset X$, we can obtain a bundle on $S$ simply by restricting the projection to $\pi^{-1}(S) \subset E$ and restricting the frames. (This is a special case of another operation called *pullback*, perhaps the most important bundle operation, which we will save for another class.)

32.2. **Subbundles.** A subspace $D \subset E$ is called a **subbundle** if $D_x \subset E_x$ is a linear subspace of dimension $s$ for each $x \in X$, and with the induced operations, $(D, X, \pi|_D)$ is a vector bundle.

The situation is closely analogous to that of submanifolds, only simpler because the inverse function theorem is not involved.

**Lemma 32.5.** *A subspace $D \subset E$ is a subbundle of rank $s$ if and only if there exist local frames for $E$ of the form*

$$\{e_1, \dots, e_s, e_{s+1}, \dots, e_r\},$$

*where $\{e_1, \dots, e_s\}$ is a local frame for $D$.*

*Proof.* ($\Rightarrow$) Let $x_0 \in X$. By assumption, there exists a local frame for $D$, $\{e_1, \dots, e_s\}$ as well as a local frame $\bar{e}_1, \dots, \bar{e}_r$ for $E$ over the same neighborhood $U \ni x_0$. Choose $i_1, \dots, i_{r-s}$ such that the collection

$$e_1(x_0), \dots, e_s(x_0), \bar{e}_{i_1}(x_0), \dots, \bar{e}_{i_{r-s}}(x_0)$$

is a basis for $E_{x_0}$. Then these are also linearly independent on a smaller neighborhood $U_0 \subset U$ with $U_0 \ni x_0$ (since this amounts to nonvanishing of a determinant).

($\Leftarrow$) This direction is trivial. □

**Examples 32.6.**

- Let $t_1, \dots, t_s \subset \Gamma(E)$ be any set of global sections such that $t_1(x), \dots t_s(x)$ is linearly independent for each $x \in X$. Then $\operatorname{Span}\{t_1, \dots, t_s\}$ is a subbundle.
- Let $S \subset M$ be a smooth submanifold. Then $TS \subset TM|_S$ is a subbundle (HW).

**Proposition 32.7.** *Suppose $\rho : E \to F$ is a bundle morphism of constant rank. Then* $\ker \rho \subset E$ *and* $\operatorname{im} \rho \subset F$ *are subbundles.*

*Proof.* We prove the statement about the kernel; the statement about the image is similar.

Let $x_0 \in X$. We can choose frames for $E$ and $F$ near $x_0$ such that the matrix of $\rho$ takes the form

$$\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}$$

with $A(x_0) = \mathbf{1}, C(x_0) = B(x_0) = D(x_0) = 0$. Note that row operations do not change the kernel. Since $A(x)$ remains invertible in a neighborhood, we can do continuous row operations to make $C(x) \equiv 0$. But since the rank is constant, we must also have $D(x) \equiv 0$. So, after row operations, our matrix takes the form

$$\begin{pmatrix} A(x) & B(x) \\ 0 & 0 \end{pmatrix}$$

A frame for the kernel is given by the columns of the matrix

$$\begin{pmatrix} A^{-1}(x)B(x) \\ -\mathbf{1} \end{pmatrix}.$$

□

**Remark 32.8.** For a constant-rank morphism, one can also define the **cokernel** (or quotient) bundle by similar local considerations, or simply by defining

$$F/\rho(E) := \left(\ker(\rho^* : F^* \to E^*)\right)^*.$$

One important instance of this is the **normal bundle** to a submanifold $S \subset M$, defined to be the quotient of the inclusion $TS \to TM|_S$. We do not have time to discuss these in greater detail.

## 33. Orientation of vector bundles (Mon 11/11-Wed 11/13)

**Definition 33.1.** Let $E \to X$ be a real vector bundle of rank $r$ ($K = \mathbb{R}$). We say that $E \to X$ is **orientable** if the determinant line bundle $\det(E)$ is trivial. (Recall that $\det(E) = \Lambda^r E$ by definition.) Equivalently, $\det(E^*) = \Lambda^r E^* \cong (\Lambda^r E)^*$ is trivial.

Since $\det(E^*)$ is a line bundle, this is equivalent to admitting a global nonvanishing section. An **orientation** of $E$ is an equivalence class of global nonvanishing sections, where $\omega \sim \eta$ iff there exists a positive continuous function $f \in C^0(X)$ such that $\omega = f\eta$.

**Lemma 33.2.** *If $X$ is connected and $E \to X$ is orientable, then there are exactly two orientations of $E$.*

*Proof.* Let $[\omega]$ and $[\eta]$ be two orientations. Since $\omega$ is a global frame for $\det(E)^*$, there exists a nowhere-vanishing function $f \in C^0(X)$ such that $\omega = f \cdot \eta$. Since $X$ is connected, we either have $f(x) > 0$ or $f(x) < 0$ for all $x \in X$. In the first case, $[\omega] = [\eta]$, while in the second case they define different orientations. □

**Examples 33.3.**

- The Möbius strip $S \to S^1$ is nontrivial (HW) and itself of rank one, therefore not orientable. (In fact its tangent bundle is also not orientable, as we shall see below.)

- The tangent bundle $TS^2 \to S^2$ is orientable. Consider the section $\omega \in \Gamma\left((TS^2)^{\otimes 2}\right)^*$ defined by

$$v \otimes w \mapsto \frac{1}{2}\langle x \times v, w\rangle = \frac{1}{2}\begin{vmatrix} x & v & w \end{vmatrix}.$$

  Notice that this is antisymmetric in $v$ and $w$, because

$$\begin{vmatrix} x & v & w \end{vmatrix} = -\begin{vmatrix} x & w & v \end{vmatrix},$$

since permuting the columns negates the determinant. Restricting the above to a functional on $\Lambda^2 TS^2 \subset (TS^2)^{\otimes 2}$, we obtain

$$\omega : \Lambda^2 TS^2 \to \underline{\mathbb{R}}$$
$$v \wedge w = v \otimes w - w \otimes v \mapsto \langle x \times v, w \rangle ,$$

which vanishes nowhere.

Using the determinant on $\mathbb{R}^{n+1}$, one can show in the same way that $TS^n$ is orientable for any $n$.

- If $M$ is a parallelizable manifold then $TM$ is orientable (since trivial).
- We will see below that any bundle on a simply-connected, paracompact space is orientable (indeed, all real line bundles on a simply-connected space are trivial).

The reason for calling $[\omega]$ an "orientation" has to do with the following.

**Definition 33.4.** Suppose that $E$ is orientable and fix an orientation $[\omega]$ . A frame $\{e_1, \dots, e_r\}$ on $U$ is (positively) **oriented** if

$$\omega(e_1(x), \dots, e_r(x)) > 0$$

for all $x \in U$. If $\omega(e_1(x), \dots, e_r(x)) < 0$ then $\{e_1, \dots, e_r\}$ is said to be *negatively oriented*. Notice that the definition is independent of the representative of $[\omega]$ .

**Proposition 33.5.** *Suppose that $X$ is paracompact and locally connected. Then $E \to X$ is orientable if and only if there exists a system of local frames for $E$ whose transition functions all belong to* $\mathrm{GL}^+(r, \mathbb{R})$.

*Proof.* ($\Rightarrow$) Let $\omega$ represent the orientation and let $\{(U_a, e^a_\alpha)\}$ be any system of trivializations. Since $X$ is locally connected, by splitting up the open sets, we can assume wlog that each open set $U_a$ is connected. By the previous Lemma, we have either $\omega(e^a_1, \dots, e^a_r) > 0$ or $\omega(e^a_1, \dots, e^a_r) < 0$ for each $a$. In the first case, do nothing, and in the second case, replace $(e^a_1, \dots, e^a_r)$ by $(e^a_2, e^a_1, e^a_3, \dots, e^a_r)$. After this change, we have $\omega(e^a_1, \dots, e^a_r) > 0$ for each $a$.

Now, on $U_a \cap U_b$, we have

$$\omega(e^a_1, \dots, e^a_r) = \omega(\sigma_{ab} e^b_1, \dots, \sigma_{ab} e^b_r) = \det \sigma_{ab}\, \omega(e^b_1, \dots, e^b_r).$$

Since both $\omega(e^a_1, \dots, e^a_r) > 0$ and $\omega(e^b_1, \dots, e^b_r) > 0$, we must have $\det \sigma_{ab} > 0$, so $\sigma_{ab} \in \mathrm{GL}^+(r, \mathbb{R})$, as desired.

($\Leftarrow$) Let $\{e^a_\alpha\}$ be the frames as in the statement, and let $\{e^\beta_a\}$ be the dual frames. We can define a nonvanishing section $\omega_a \in \Gamma(U_a, \det(E)^*)$ by

$$\omega_a = e^1_a \wedge \cdots \wedge e^r_a.$$

Notice that on $U_a \cap U_b$, we have

$$\begin{aligned} \omega_b(e^a_1, \dots, e^a_r) &= \omega_b(\sigma_{ab} e^a_1, \dots, \sigma_{ab} e^a_r) \quad \text{(no sum on } a, b) \\ &= \det \sigma_{ab}. \end{aligned}$$

Let $\rho_a$ be a partition of unity subordinate to $\{U_a\}$, and let

$$\omega = \sum_a \rho_a \omega_a.$$

For each $a$, we have $\omega(e_1^a, \dots, e_r^a) = \sum_b \rho_b \omega_b(e_1^a, \dots, e_r^a) = \sum_b \rho_b \det \sigma_{ab} > 0$. □

**Corollary 33.6.** *The underlying real bundle of a complex line bundle is orientable.*

*Proof.* By the previous proposition, we just need to check that in the underlying real coordinate frames of a system of complex frames, the transition functions have positive determinant. To obtain real frames from complex frames, we take $\{1, i\}$ as our basis for $\mathbb{C} \cong \mathbb{R}^2$. Let $\sigma = \sigma_{ab} = a + bi \in \mathbb{C}^\times$. Then

$$(\sigma \cdot -) : \mathbb{C}^\times \to \mathbb{C}^\times$$

acts by

$$(a + bi) \cdot 1 = a + bi, \qquad (a + bi) \cdot i = -b + ai.$$

So the corresponding matrix is

$$\sigma_{\mathbb{R}} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{33.1}$$

We have $\det \sigma_{\mathbb{R}} = a^2 + b^2 > 0$. □

**Remark 33.7.** In fact, the underlying real bundle of a complex vector bundle of any rank is orientable. This can be seen from the formula $\det_{\mathbb{R}} \sigma_{\mathbb{R}} = |\det_{\mathbb{C}} \sigma|^2$.

## 34. Example: $\mathscr{O}(n) \to \mathbb{CP}^1$ (Wed 11/13)

Let us discuss the example of complex line bundles on $\mathbb{CP}^1$ in more detail. We first give a geometric proof of the following fact:

**Proposition 34.1.** *$\mathscr{O}(-1) \to \mathbb{CP}^1$ is nontrivial.*

*Proof.* Suppose, for contradiction, that there exists a nowhere-vanishing global section

$$s : \mathbb{CP}^1 \to \mathscr{O}(-1).$$

We can post-compose with the inclusion map into $\mathbb{CP}^1 \times \mathbb{C}^2$ in the definition of $\mathscr{O}(-1)$, followed by projection to the second factor:

$$\mathbb{CP}^1 \xrightarrow{s} \mathscr{O}(-1) \to \mathbb{CP}^1 \times \mathbb{C}^2 \to \mathbb{C}^2.$$

Since $s$ is nowhere-vanishing, the image of this composition lies in $\mathbb{C}^2 \setminus \{0\}$. We then post-compose with the projection map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$. In this way, we obtain a sequence of continuous maps

$$\mathbb{CP}^1 \to \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1.$$

Since $s$ was a section of the tautological bundle, the composition must be the identity map. Taking second homology, we obtain a sequence

$$H_2(\mathbb{CP}^1, \mathbb{Z}) \to H_2(\mathbb{C}^2 \setminus \{0\}, \mathbb{Z}) \to H_2(\mathbb{CP}^1, \mathbb{Z})$$

whose composition is again the identity. However, $\mathbb{C}^2 \setminus \{0\}$ is homotopy equivalent to $S^3$, so its second homology vanishes, whereas $H_2(\mathbb{CP}^1) = H_2(S^2) = \mathbb{Z}$. The above sequence is

$$\mathbb{Z} \to 0 \to \mathbb{Z},$$

so the composition is zero. We have reached a contradiction. □

Next, we want to examine the bundles $\mathscr{O}(n)$, given in Definition 32.3, over $\mathbb{CP}^1$. This is most readily accomplished using the frame-based approach. We will rely on the following result.

**Lemma 34.2.** *Suppose that $\mathscr{L}$ and $\mathscr{M}$ are complex line bundles over $\mathbb{CP}^1$, each trivial in the stereographic charts. Then $\mathscr{L} \cong \mathscr{M}$ (if and) only if their respective transition functions $\sigma, \mu : \mathbb{C}^\times \to \mathbb{C}^\times$ are homotopic.*

*Proof.* We will only prove the "only if" direction, since the "if" direction requires a bit more care.

Assume that $\mathscr{L} \cong \mathscr{M}$. We use the characterization of isomorphism given by Proposition 31.4. Let $U_\pm = \mathbb{CP}^1 \setminus p_\mp$ be stereographic charts centered at antipodal points, and let $x_\pm \in \mathbb{C}$ be the corresponding coordinates. There exist $\tau_\pm \in C^0(U_\pm, \mathbb{C}^\times)$ such that

$$\sigma = \tau_-^{-1} \mu \tau_+.$$

Consider the continuous functions from $[0,1] \times U_\pm \to \mathbb{C}^\times$ defined by

$$\tau_\pm^t = \tau_\pm(t x_\pm)$$

For $t = 1$, we have

$$\tau_\pm^1(x_\pm) = \tau_\pm(x_\pm),$$

while for $t = 0$, we have

$$\tau_\pm^0(x_\pm) \equiv \tau_\pm(p_\pm).$$

We can use these to make a homotopy

$$\begin{aligned} \sigma = \tau_-^{-1} \mu \tau_+ &= (\tau_-^1)^{-1} \mu \tau_+^1 \\ &\sim (\tau_-^0)^{-1} \mu \tau_+^0 \\ &= \tau_-(p_-)^{-1} \mu \tau_+(p_+). \end{aligned}$$

Since $\mathbb{C}^\times$ is path-connected, we can then move $\tau_+(p_+)$ and $\tau_-(p_-)$ along paths to the identity. This gives us a homotopy

$$\sigma \sim \tau_-(p_-)^{-1} \mu \tau_+(p_+) \sim \mu,$$

as desired. □

**Proposition 34.3.** $\mathscr{O}(m) \cong \mathscr{O}(n) \Leftrightarrow m = n$.

*Proof.* From (33.1), the transition function of $\mathscr{O}(n)$ is given by

$$z^n \cdot = r^n \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}.$$

Restricting to the equator $r = 1$, we see that $z^n : S^1 \to S^1$ represents the homotopy class $[n] \in \pi_1(S^1) \cong \mathbb{Z}$. So for $m \neq n$, the transition functions of $\mathscr{O}(n)$ and $\mathscr{O}(m)$ belong to different homotopy classes. By the previous Lemma, they cannot be isomorphic. □

**Remark 34.4.** The Lemma is a rudimentary form of the "clutching construction" which produces and classifies all bundles of a given rank on $S^n$. The Proposition is also much more general: it turns out that $\{\mathscr{O}(m)\}_{m\in\mathbb{Z}}$ are the only complex line bundles that exist on $\mathbb{CP}^n$, including when holomorphic structure is taken into account.

# Part 7. Tensors and differential forms

## 35. Smooth bundles, tensor characterization lemma (Fri 11/15)

**Definition 35.1.** Let $M$ be a smooth manifold and $\pi : E \to M$ a vector bundle. We say that $E$ is **smooth** if $E$ is also a smooth manifold, $\pi$ is a submersion, the operations + and $\cdot$ are smooth on $E_x$ for each $x \in M$, and there exist smooth local frames such that the maps (30.1) are diffeomorphisms.

All of the results of the last chapter work in the category of smooth bundles and smooth morphisms. Henceforth we will work only with smooth bundles, smooth sections, smooth trivializations, etc. The notation for space of sections will change to mean:

$$\Gamma(U, E) = \text{space of } \textit{smooth} \text{ sections of } E \text{ over } U.$$

Meanwhile, the results of this section (and some later in the chapter) also work perfectly well for topological bundles, but it's time to shift our focus back to smooth manifolds.

**Example 35.2.** As remarked above, for a smooth manifold $M$, the tangent bundle $TM \to M$ is a smooth vector bundle.

We now make the following essentially trivial observations. Given a section $s \in \Gamma(U, E)$ and a smooth function $f \in C^\infty(U)$, we can multiply these together to obtain another section $f \cdot s \in \Gamma(U, E)$ given by

$$(f \cdot s)(x) = f(x)s(x).$$

Since $fg \cdot s = f \cdot (g \cdot s)$, $(f + g) \cdot s = f \cdot s + g \cdot s$, and $f \cdot (s + t) = f \cdot s + f \cdot t$, this makes $\Gamma(U, E)$ into a **module** over the ring $C^\infty(U)$.

Similarly, given a section $\alpha \in \Gamma(U, E^*)$, we can obtain a map

$$\begin{aligned} \Gamma(U, E) &\to C^\infty(U) \\ s &\mapsto \alpha(s), \end{aligned}$$

where $\alpha(s)(x) = \alpha(x)(s(x))$. Observe that this map is **$C^\infty(U)$-linear**, in other words, a module homomorphism:

$$\begin{aligned} \alpha(f \cdot s)(x) &= \alpha(x)\left(f(x)s(x)\right) \\ &= f(x)\alpha(x)(s(x)) \\ &= (f \cdot \alpha(s))(x). \end{aligned}$$

Having made these trivial observations, we can prove the following not-quite-trivial result.

**Lemma 35.3.** *For any open set $U \subset M$, the above map*

$$\Gamma(U, E^*) \to \operatorname{Hom}_{C^\infty(U)}\left(\Gamma(U, E), C^\infty(U)\right)$$

*is an isomorphism.*

*Proof. Injectivity.* Let $\beta \in \Gamma(U, E^*)$ and assume that $\beta \neq 0$. Choose $x_0 \in U$ such that $\beta(x_0) \neq 0$. Then there exists $v \in E_{x_0}$ such that $\beta(x_0)(v) \neq 0$. Choose $V \subset U$ over which $E$ is trivial, so that we can extend $v$ to a section over $s \in \Gamma(U, E)$ with $s(x_0) = v$. Let $\rho$ be a bump function for $\{x_0\} \subset V$. We have

$$\beta(\rho \cdot s)(x_0) = \beta(x_0)(\rho(x_0)s(x_0)) = 1 \cdot \beta(x_0)(v) \neq 0.$$

This shows that the homomorphism represented by $\beta$ is nonzero.
*Surjectivity.* Suppose first that $U$ is small enough that $E$ is trivial over $U$. Let $\{e_\alpha\}$ be a local frame over $U$. Given $\tau \in \mathrm{Hom}_{C^\infty(U)}(\Gamma(U,E), C^\infty(U))$, take

$$\beta = \tau(e_\alpha)e^\alpha.$$

We then have

$$\begin{aligned}\beta(s) &= \tau(e_\alpha)e^\alpha(s)\\ &= \tau(e_\alpha)s^\beta \delta^\alpha{}_\beta\\ &= s^\alpha \tau(e_\alpha)\\ &= \tau(s^\alpha e_\alpha) = \tau(s),\end{aligned}$$

so $\beta$ agrees with $\tau$, as desired.

For the general case, suppose that $U = \cup U_a$ with $E|_{U_a}$ trivial for each $a$ and $\{U_a\}$ locally finite. We can assume wlog that $U_a \Subset \hat{U}_a$ on which $E|_{\hat{U}_a}$ is still trivial. Let $\{e^a_\alpha\}$ be frames over $\hat{U}_a$, and $\{e^\alpha_a\}$ the dual frames. Multiplying by a bump functions for $U_a \Subset \hat{U}_a$, we can extend each $e^\alpha_a$ to a section $\tilde{e}^\alpha_a \in \Gamma(U, E^*)$ such that $\tilde{e}^\alpha_a(x) = e^\alpha_a(x)$ for $x \in U_a$.

Choose a partition of unity $\{\rho_a\}$ subordinate to $\{U_a\}$, and let

$$\beta = \sum_{a,\alpha} \tau(\rho_a e^a_\alpha)\tilde{e}^\alpha_a.$$

For $s \in \Gamma(U,E)$, we have

$$\begin{aligned}\beta(s) &= \sum_{a,\alpha} \tau(\rho_a e^a_\alpha)\tilde{e}^\alpha_a(s)\\ &= \sum_{a,\alpha} \tilde{s}^\alpha_a \tau\left(\rho_a e^a_\alpha\right),\end{aligned}$$

where $\tilde{s}^\alpha_a := \tilde{e}^\alpha_a(s)$. This satisfies $\tilde{s}^\alpha_a = s^\alpha_a$ on $U_a$, so that

$$\rho_a \tilde{s}^\alpha_a = \rho_a s^\alpha_a.$$

Applying $C^\infty(U)$-linearity of $\tau$, we get[14]

$$\begin{aligned}\beta(s) = \sum_{a,\alpha} \tau(\rho_a \tilde{s}^\alpha_a e^a_\alpha) &= \sum_a \sum_\alpha \tau\left(\rho_a s^\alpha_a e^a_\alpha\right)\\ &= \sum_a \tau\left(\sum_\alpha \rho_a s^\alpha_a e^a_\alpha\right)\\ &= \sum_a \tau\left(\rho_a s\right)\\ &= \sum_a \rho_a \tau\left(s\right)\\ &= \left(\sum_a \rho_a\right)\tau\left(s\right) = \tau\left(s\right).\end{aligned}$$

□

[14]Thanks to Haran Mouli for pointing out after class how to save this argument.

We now wish to generalize the above discussion to multilinear functions. Let $E_1, \ldots, E_m$ be smooth vector bundles over $M$. A section $\beta \in \Gamma(E_1^* \otimes \cdots \otimes E_m^*)$, as above, induces a $C^\infty(M)$**-multilinear** map

$$\begin{aligned} \Gamma(E_1) \times \cdots \times \Gamma(E_m) &\to C^\infty(M) \\ (s_1, \ldots, s_m) &\mapsto \beta(x)(s_1(x), \ldots, s_m(x)). \end{aligned}$$

We state the following lemma for global sections over $M$, but of course it remains true after replacing $M$ by any open subset (submanifold) $U \subset M$.

**Lemma 35.4** (Tensor characterization lemma)**.** *Each $C^\infty(M)$-multilinear map as above is induced by a unique section $\beta \in \Gamma(E_1^* \otimes \cdots \otimes E_m^*)$.*

*Proof.* The injectivity can be shown along very similar lines to the previous proof.

To show surjectivity, we use induction. The case $m = 1$ is the previous Lemma. Suppose that the Lemma has been proven for $m-1$, and let $\tau$ be a $C^\infty$-multilinear function on $\Gamma(E_1) \times \cdots \times \Gamma(E_m)$.

As in the previous proof, let $\{U_a\}$ be a locally finite cover such that there exists a system of frames $e_\alpha^a$ for $E_1$ over $U_a$, with dual frames $e_a^\alpha$, and extend these to global sections $\tilde{e}_a^\alpha \in \Gamma(E_1^*)$. Also fix a partition of unity $\{\rho_a\}$ subordinate to $\{U_a\}$.

Observe that given any section $s_1 \in \Gamma(E_1)$, the function $\tau(s_1, \cdot, \ldots, \cdot)$ is multilinear in $m-1$ arguments. Let

$$\tau_\alpha^a \in \Gamma(E_2^* \otimes \cdots \otimes E_m^*)$$

denote the section which corresponds, by the induction hypothesis, to

$$\tau(\rho_a e_\alpha^a, \cdot, \ldots, \cdot) \quad \text{(no sum on } a).$$

Let

$$\beta = \sum_{a,\alpha} \tilde{e}_a^\alpha \otimes \tau_\alpha^a.$$

We now argue as in the last proof. For $s_i \in \Gamma(U, E_i)$, we have

$$\begin{aligned} \beta(s_1, \ldots, s_m) &= \sum_{a,\alpha} \tilde{e}_a^\alpha(s_1) \tau_\alpha^a (s_2, \ldots, s_n) \\ &= \sum_{a,\alpha} \widetilde{(s_1)}_a^\alpha \tau \left(\rho_a e_\alpha^a, s_1, \ldots, s_n\right), \end{aligned}$$

where $\widetilde{(s_1)}_a^\alpha := \tilde{e}_a^\alpha(s_1)$. This satisfies $\widetilde{(s_1)}_a^\alpha = (s_1)_a^\alpha$ on $U_a$, so that

$$\rho_a \widetilde{(s_1)}_a^\alpha = \rho_a (s_1)_a^\alpha.$$

Applying $C^\infty$-multilinearity of $\tau$, we get

$$\begin{aligned}\beta(s_1,\dots,s_m) &= \sum_{a,\alpha}\widetilde{(s_1)}_a^\alpha \tau\left(\rho_a e^a_\alpha, s_2,\dots,s_n\right)\\ &= \sum_a \tau\left(\sum_\alpha \rho_a \widetilde{(s_1)}_a^\alpha e^a_\alpha, s_2,\dots,s_n\right)\\ &= \sum_a \tau\left(\rho_a s_1, s_2,\dots,s_n\right)\\ &= \sum_a \rho_a\tau\left(s_1, s_2,\dots,s_n\right)\\ &= \tau\left(s_1,\dots,s_m\right).\end{aligned}$$

□

## 36. Tensors on smooth manifolds, pullback, Lie derivative (Mon 11/18)

36.1. **The definition.** We now specialize to the case that $E_i = TM$ or $T^*M$ for each $i$.

The space of $(k,\ell)$**-tensors** is denoted by

$$T^{(k,\ell)}(M) := \Gamma(\overbrace{T^*M\otimes\cdots\otimes T^*M}^{k}\otimes\overbrace{TM\otimes\cdots\otimes TM}^{\ell}).$$

A tensor of type $(k,\ell)$ is called a **covariant** $k$-tensor, and a tensor of type $(0,\ell)$ is called a **contravariant** $\ell$-tensor. We have $T^{(0,1)}(M) = \mathscr{X}(M)$, so a contravariant 1-tensor is a vector field, while an element of $T^{(1,0)}(M) =: \Omega^1(M)$ is a covector field. In local coordinates, an element $A \in T^{(k,\ell)}(M)$ is given by

$$A \overset{loc}{=} A_{i_1\cdots i_k}{}^{j_1\cdots j_\ell} dx^{i_1}\otimes\cdots\otimes dx^{i_k}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_k}}.$$

By the tensor characterization lemma 35.4, an element $A\in T^{(k,\ell)}$ is equivalent to a multilinear map

$$\begin{aligned}A : \overbrace{\mathscr{X}(M)\times\cdots\times\mathscr{X}(M)}^{k}\times\overbrace{\Omega^1(M)\times\cdots\times\Omega^1(M)}^{\ell} &\to C^\infty(M)\\ (X_1,\dots,X_k,\alpha_1,\dots,\alpha_\ell) &\mapsto A(X_1,\dots,X_k,\alpha_1,\dots,\alpha_\ell)\\ &\overset{loc}{=} A_{i_1\cdots i_k}{}^{j_1\cdots j_\ell}(X_1)^{i_1}\cdots(X_k)^{i_k}(\alpha_1)_{j_1}\cdots(\alpha_\ell)_{j_\ell}.\end{aligned}$$

Given two tensors $A\in T^{(k_1,\ell_1)}$ and $B\in T^{(k_2,\ell_2)}$, we can define the **tensor product**

$$A\otimes B\in T^{(k_1+k_2,\ell_1+\ell_2)}$$

simply by multiplication of $C^\infty$-multilinear functions, as in §29.2.1. We can also **contract** the $m$'th covariant index with the $n$'th contravariant index:

$$A\in T^{(k,\ell)}(M)\mapsto \operatorname{Tr}_{m,n}A\in T^{(k-1,\ell-1)}(M)$$

$$A_{i_1\cdots i_k}{}^{j_1\cdots j_\ell}\overset{loc}{\mapsto} A_{i_1\cdots i\cdots i_{k-1}}{}^{j_1\cdots i\cdots j_\ell}.$$

These operations work in just the same way as they do for general vector bundles (all based on the same operations on vector spaces).

We will now specialize to covariant tensors, since these will be the most important ones going forward.

36.2. **Pullback of covariant tensors.** Let $F : M \to N$ be a smooth map and $A \in T^{(k,0)}(N)$ a covariant $k$-tensor on $N$. The **pullback** of $A$ by $F$ is defined by

$$F^*(A)(X_1, \ldots, X_k)(p) := A(dF_p(X_1), \ldots dF_p(X_k)).$$

This is a smooth $(k,0)$-tensor on $M$. If $F(x) = (y^j(x^i))$ in local coordinates, we have

$$(F^*A)_{i_1 \cdots i_k} = \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_k}}{\partial x^{i_k}} \left(A_{j_1 \cdots j_k} \circ F\right).$$

Note that if $F$ is a diffeomorphism then one can also pull back (or push forward) contravariant or mixed tensors, see Prop 18.2 above.

**Example 36.1.** Take $M$ and $N$ to be domains in Euclidean space and $k = 1$, so $A = \alpha_j(y)dy^j$. We have

$$F^*(A) = \alpha_j(y(x))\frac{\partial y^j}{\partial x^i}dx^i.$$

We now state the following properties of pullback; you should think through the proofs as an exercise.

**Proposition 36.2.** *(a)* $F^*(f \cdot A) = (f \circ F)F^*A$
*(b)* $F^*(A \otimes B) = F^*A \otimes F^*B$
*(c)* $F^*(A + B) = F^*A + F^*B$
*(d)* $(G \circ F)^*A = F^*G^*A$.

36.3. **Lie derivative on tensors.** Let $V \in \mathscr{X}(M)$ and $\theta_t$ be the flow of $V$. Given $A \in T^{(k,0)}(M)$, the **Lie derivative** of $A$ is defined by

$$(\mathscr{L}_V A)_p = \left.\frac{d}{dt}\right|_{t=0} (\theta_t^* A)_p = \lim_{t \to 0} \frac{(d\theta_t)_p^*(A_{\theta_t(p)}) - A_p}{t}.$$

This is a smooth element of $T^{(k,0)}(M)$.

*Proof.* We have $\theta_t^* A$ smooth in all variables and $\theta_0^* A = A$, so the result is smooth. □

**Proposition 36.3** (Properties of the Lie derivative of a covariant tensor)**.** *(a) For* $k = 0$, *we have* $\mathscr{L}_V f = V(f)$
*(b)* $\mathscr{L}_V(fA) = (\mathscr{L}_V f)A + f\mathscr{L}_V A$
*(c)* $\mathscr{L}_V (A \otimes B) = \mathscr{L}_V A \otimes B + A \otimes \mathscr{L}_V B$
*(d)* $V(A(X_1, \ldots, X_k)) = (\mathscr{L}_V A)(X_1, \ldots, X_k) + A(\mathscr{L}_V X_1, \ldots, X_k) + \cdots + A(X_1, \ldots, \mathscr{L}_V X_k)$.

*Proof.* (a) was proved in Remark 21.4 above (since $(\theta_{-1})_* = \theta_t^*$ on functions).

(b-d) As in the proof of Theorem 21.2, we can assume without loss of generality that $p$ is a regular point and $V = \frac{\partial}{\partial u^1}$ in some coordinates $u^1, \ldots, u^n$ near $p$. Then the flow of $V$ is translation by $t$ in the first variable, so

$$(\mathscr{L}_V A)_{i_1 \cdots i_k} = \frac{\partial A_{i_1 \cdots i_k}}{\partial u_1}.$$

The stated properties all follow from the ordinary Leibniz rule. □

**Corollary 36.4.** $(\mathscr{L}_V A)(X_1, \ldots, X_k) = V(A(X_1, \ldots, X_k)) - A([V, X_1], X_2, \ldots, X_k) - \cdots - A(X_1, \ldots, [V, X_k])$.

**Corollary 36.5.**

$$\mathscr{L}_V(df) = d(\mathscr{L}_V f)$$

*Proof.*

$$\begin{aligned}
(\mathscr{L}_V df)(X) &= V(df(X)) - df([V, X]) \\
&= V(X(f)) - V(X(f)) + X(V(f)) \\
&= d(Vf)(X) \\
&= d(\mathscr{L}_V f)(X).
\end{aligned}$$

□

**Remark 36.6.** This result will generalize below to differential forms.

**Example 36.7.** Let $A = A_{ij} dx^i \otimes dx^j \in T^{(2,0)}(M)$. Let $V = V^i \frac{\partial}{\partial x^i}$, so that $\mathscr{L}_V(dx^i) = d(V^i)$ by the previous corollary. We have

$$\begin{aligned}
\mathscr{L}_V A &= V(A_{ij}) dx^i \otimes dx^j + A_{ij} dV^i \otimes dx^j + A_{ij} dx^i \otimes dV^j \\
&= \left( V(A_{ij}) + A_{kj} \frac{\partial V^k}{\partial x^i} + A_{ik} \frac{\partial V^k}{\partial x^j} \right) dx^i \otimes dx^j.
\end{aligned}$$

A similar formula exists in general.

Last, we want to give the generalization of Proposition 22.2.

**Lemma 36.8.** $\left.\frac{d}{dt}\right|_{t=t_0} (\theta_t)^* A = (\theta_{t_0})^* \mathscr{L}_V A$.

*Proof.* The proof is the same as that of Lemma 22.1. □

**Proposition 36.9.** *$A$ is invariant under $\theta_t$ if and only if $\mathscr{L}_V A = 0$.*

## 37. Differential forms (Mon 11/18-Wed 11/20)

37.1. **The definition.** A **differential $k$-form** is an alternating covariant $k$-tensor. We let

$$\Omega^k(M) := \Gamma(\Lambda^k T^* M) \subset \Gamma((T^* M)^{\otimes k}) = T^{(k,0)}(M)$$

denote the space of $k$-forms, so $\omega \in \Omega^k(M)$ satisfies

$$\omega(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k) = -\omega(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_k)$$

for each $1 < i \neq j < k$.

The **wedge product** $\wedge : \Omega^k \times \Omega^\ell \to \Omega^{k+\ell}$ on differential forms is inherited from the wedge product on vector spaces described above, and can be written:

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\ell!}\mathrm{Alt}\,(\omega \otimes \eta).$$

This is $C^\infty$-bilinear and associative, and satisfies

$$\omega \wedge \eta = (-1)^{k\ell}\eta \wedge \omega,$$

where the factor is the sign of the permutation which takes the first $k$ arguments to the last $k$ arguments. As we explained briefly in the section on linear algebra, the coefficient is rigged so that

$$dx^1 \wedge \cdots \wedge dx^k = \sum_{\sigma \in S_k} \mathrm{sgn}\sigma\, dx^{\sigma(1)} \otimes \cdots \otimes dx^{\sigma(k)}, \tag{37.1}$$

which gives the convenient normalization

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right) = 1.$$

More generally, for a multi-index $(i_1, \ldots, i_k)$ with distinct elements, we have

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \begin{cases} \mathrm{sgn}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} & \{i_1, \ldots, i_k\} = \{j_1, \ldots, j_k\} \\ 0 & \text{otherwise.} \end{cases}$$

**Example 37.1.**

$$y dx \wedge dz\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}\right) = -y,$$

$$y dx \wedge dz\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0.$$

Here is a useful general formula:

**Lemma 37.2.** *Let $\omega^1, \ldots, \omega^k \in \Omega^1(M)$ and $X_1, \ldots, X_k \in \mathscr{X}(M)$. We have*

$$\omega^1 \wedge \cdots \wedge \omega^k(X_1, \ldots, X_k) = \det\left(\omega^i(X_j)\right).$$

*Proof.* As in (37.1), we have

$$\omega^1 \wedge \cdots \wedge \omega^k = \sum_{\sigma \in S_n} \mathrm{sgn}\,\sigma\, \omega^{\sigma(1)} \otimes \cdots \otimes \omega^{\sigma(k)}.$$

We get

$$\omega^1 \wedge \cdots \wedge \omega^k(X_1, \ldots, X_k) = \sum_{\sigma \in S_k} \mathrm{sgn}\,\sigma\, \omega^{\sigma(1)}(X_1) \cdots \omega^{\sigma(k)}(X_k),$$

which is the desired expression. □

We can summarize this discussion as follows.

**Proposition 37.3.** *Let* $\{e^i\}_{i=1}^n$ *be a local coframe, i.e., a local frame for* $T^*M$*. Then*

$$\{e^{i_1} \wedge \cdots \wedge e^{i_k} \mid i_1 < \cdots < i_k\}$$

*is a local frame for* $\Lambda^k T^*M$*. In particular, if* $\{x^i\}$ *are any local coordinates, the set*

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} \mid i_1 < \cdots < i_k\}$$

*forms a local frame for* $\Lambda^k T^*M$*, in which any differential form can be uniquely written as*

$$\omega \overset{loc}{=} \sum_I{}' \omega_I dx^I,$$

*where* $I = (i_1, \ldots, i_k)$ *is a multi-index,* $\omega_I = \omega_{i_1 \cdots i_k}$*,* $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$*, and* $\sum_I' = \sum_{i_1 < \cdots < i_k}$ *denotes the sum over strictly increasing multi-indices.*

37.2. **Interior product.** Another important operation is the **interior product**

$$\iota : \mathscr{X}(M) \times \Omega^k(M) \to \Omega^{k-1}(M)$$
$$(X, \omega) \mapsto \iota_X \omega = X \lrcorner\, \omega = \omega(X, -, \ldots, -).$$

**Example 37.4.** $\iota_{x \frac{\partial}{\partial y}}(dx \wedge dy) = -x dx$.

**Proposition 37.5** (Properties of $\iota_X$)**.** *(a)* $\iota_X \circ \iota_X = 0$

*(b)* $\iota_X(\omega \wedge \eta) = (\iota_X \omega) \wedge \eta + (-1)^k \omega \wedge \iota_X \eta$.

*Proof.* (a) is obvious from the alternating property.

For (b), it suffices by bilinearity of both sides in $\omega$ and $\eta$ to consider the case that both $\omega$ and $\eta$ are decomposable. Let $\omega = \omega^1 \wedge \cdots \wedge \omega^k$ and $\eta = \omega^{k+1} \wedge \cdots \wedge \omega^m$, so that $\omega \wedge \eta = \omega^1 \wedge \cdots \wedge \omega^m$. Inspection shows that the desired formula follows from the general formula

$$\iota_X \left(\omega^1 \wedge \cdots \wedge \omega^m\right) = \sum_{i=1}^m (-1)^{i-1} \omega^i(X) \omega^1 \wedge \cdots \wedge \hat{\omega}^i \wedge \cdots \wedge \omega^m. \tag{37.2}$$

To check (37.2), let $X_1 = X$ and let $X_2, \ldots, X_m$ be arbitrary. Let $\Omega = (\omega^i(X_j))$. By Lemma 37.2, the LHS is equal to $\det(\Omega)$. Meanwhile, again by Lemma 37.2, the RHS is equal to the expansion by minors along the first column of the determinant of $\Omega$, so the two are equal. □

**Remark 37.6.** An operation satisfying (b) is called a **graded derivation** (or sometimes an *anti-derivation*, as in Lee's book).

37.3. **Pullback of forms.** Pullback, which we know already for general covariant tensors, is a key operation on differential forms. Here are its properties:

**Proposition 37.7.** *(a)* $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$.

*(b) Let* $F : M \to N$ *and* $\omega \in \Omega^k(N)$*. Let* $\{x^i\}$ *be local coordinates on* $M$ *and* $\{y^i\}$ *be local coordinates on* $N$*. We have*

$$F^*\omega = \sum_{j_1 < \cdots < j_k} (\omega_{j_1 \cdots j_k} \circ F)\, d(y^{j_1} \circ F) \wedge \cdots \wedge d(y^{j_k} \circ F).$$

*(c) If $k = n = \dim(M) = \dim(N)$ and $\omega = u dy^1 \wedge \cdots \wedge dy^n$, then we further have*

$$F^*\omega = (u \circ F) \det(DF) dx^1 \wedge \cdots \wedge dx^n,$$

*where $DF = \left(\frac{\partial y^j}{\partial x^i}\right)$ denotes the Jacobian matrix of $y = F(x)$.*

*Proof.* $(a)$ This follows from the definition, since $F^*$ commutes with tensor product and Alt.

$(b)$ First note that $F^*(u dy^j) = (u \circ F)\frac{\partial y^j}{\partial x^i} dx^i = (u \circ F) d(y^j \circ F)$ by Example 36.1 and the chain rule. The general result follows by applying (a) to the multiple tensor product.

$(c)$ Since the top wedge powers are one-dimensional, it suffices to evaluate both sides on $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$. By (b), we get

$$\begin{aligned} LHS\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) &= (u \circ F)\frac{\partial y^1}{\partial x^{i_1}} \cdots \frac{\partial y^n}{\partial x^{i_n}} dx^{i_1} \wedge \cdots \wedge dx^{i_n}\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) \\ &= (u \circ F)\text{sgn } (i_1 \cdots i_n)\, \frac{\partial y^1}{\partial x^{i_1}} \cdots \frac{\partial y^n}{\partial x^{i_n}} \\ &= (u \circ F) \det(DF) \\ &= RHS\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right). \end{aligned}$$

□

**Example 37.8.** Let $F : (r, \theta) \mapsto (x, y) = (r\cos\theta, r\sin\theta)$ be the polar coordinate chart. By (b), we have

$$\begin{aligned} F^*(dx \wedge dy) &= d(r\cos\theta) \wedge d(r\sin\theta) \\ &= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\ &= r\left(\cos^2(\theta) + \sin^2(\theta)\right) dr \wedge d\theta \\ &= r dr \wedge d\theta. \end{aligned}$$

Meanwhile, we have the Jacobian

$$DF = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

with $\det(DF) = r$, so the pullback computed by (c) agrees with the answer from (b).

## 38. The exterior derivative (Wed 11/20-Fri 11/22)

38.1. **The case of an open set in $\mathbb{R}^n$.** Let $M = U \subset \mathbb{R}^n$ be an open subset. Then any differential form is written uniquely as

$$\omega = \sum_I{}' \omega_I dx^I$$

for $\omega_I \in C^\infty(M)$. (Recall the notation $\sum_I'$ from Proposition 37.3.) We define the **exterior derivative**

$$d\omega := \sum_I{}' d\omega_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

globally on $U$.

**Proposition 38.1** (Properties of the exterior derivative on $\mathbb{R}^n$)**.** *(a) $d$ is $\mathbb{R}$-linear, i.e. linear over constant functions.*

*(b)* $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

*(c)* $d \circ d = 0$

*(d) If $F : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ is smooth, then*

$$F^* d\omega = dF^* \omega.$$

*Proof.* (a) This is obvious.

(b) We should first check that the formula remains true after replacing $\sum_I'$ by the ordinary sum $\sum_I$ over general indices (because wedge product will not respect increasing-ness). It suffices to check that the formula is true for $\omega = u dx^I$, where $I = (i_1, \ldots, i_k)$ is not necessarily increasing. Let $\sigma \in S_k$ be the permutation such that $i_{\sigma(1)} < \cdots < i_{\sigma(k)}$ is increasing. We have

$$\omega = \operatorname{sgn} \sigma u dx^{\sigma(I)},$$

so that by definition, we have

$$\begin{aligned} d\omega &= d(\operatorname{sgn} \sigma u) \wedge dx^{\sigma(I)} \\ &= du \wedge (\operatorname{sgn} \sigma dx^{\sigma(I)}) \\ &= du \wedge (\operatorname{sgn} \sigma)^2 \, dx^I \\ &= du \wedge dx^I \end{aligned}$$

as desired.

Now, to check (b), we can compute

$$\begin{aligned} d(\omega \wedge \eta) &= d(uv dx^I \wedge dx^J) \\ &= (v du + u dv) \wedge dx^I \wedge dx^J \\ &= d\omega \wedge dx^J + (-1)^k \omega \wedge d\eta. \end{aligned}$$

(c) We first check the case $k = 0$. We have

$$\begin{aligned} d(du) &= d\left( \frac{\partial u}{\partial x^j} dx^j \right) \\ &= \frac{\partial^2 u}{\partial x^i \partial x^j} dx^i \wedge dx^j \\ &= \sum_{i<j} \left( \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j = 0 \end{aligned}$$

since second partials commute.

For general $k$, letting $\omega = \sum_I' \omega_I dx^I$, we have

$$\begin{aligned} d^2\omega &= \sum_I{}' d(d\omega_I \wedge dx^I) \\ &= \sum_I{}' d^2(\omega_I) \wedge dx^I + (-1)^{k+1} d\omega_I \wedge d(dx^I). \end{aligned}$$

The first term is zero by the $k = 0$ case. The second term contains elements of the form

$$d(dx^I) = d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = d(dx^{i_1}) \wedge \cdots \wedge dx^{i_k} - dx^{i_1} \wedge d(dx^{i_2}) \wedge \cdots \wedge dx^{i_k} + \cdots$$

which is again zero by the $k = 0$ case, so the entire expression vanishes.

(d) We may let $\omega = u dx^I$ without loss of generality. We have

$$LHS = F^*(d(u dx^I)) = F^*(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)$$

by Proposition 37.7(b). Meanwhile, we have

$$RHS = d\big((u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)\big) = d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F),$$

so the two sides agree. □

38.2. **The general case.** The following result is fundamental.

**Theorem 38.2.** *Let $M$ be a smooth manifold of dimension $n$. There exist maps $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for $k = 0, \ldots, n-1$ which are uniquely characterized by*

*(i) $d$ is linear over constant functions*

*(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$*

*(iii) $d \circ d = 0$*

*(iv) For $f \in \Omega^0(M) = C^\infty(M)$, $df$ is the ordinary differential of $f$, given by $df(X) = X(f)$.*

*Proof.* Let $\varphi : U \to \hat{U} \subset \mathbb{R}^n$ be any coordinate chart, and define

$$d\omega = \varphi^* d\left((\varphi^{-1})^*\omega\right),$$

where $d$ is the exterior derivative in $\hat{U}$ defined above. We must check that this definition is independent of the coordinate chart; let $\psi$ be another chart. We calculate:

$$\begin{aligned} \psi^* d\left((\psi^{-1})^*\omega\right) &= \psi^* d\left((\psi^{-1})^*(\varphi^{-1}\varphi)^*\omega\right) \\ &= \psi^* d\left((\psi^{-1})^*\varphi^*(\varphi^{-1})^*\omega\right) \\ &= \psi^* d\left((\varphi\psi^{-1})^*(\varphi^{-1})^*\omega\right). \end{aligned}$$

We now apply Proposition 38.1$d$ to pass the transition map through the exterior derivative (on $\mathbb{R}^n$), to obtain

$$\begin{aligned}\psi^* d\left((\psi^{-1})^*\omega\right) &= \psi^* \left(\varphi\psi^{-1}\right)^* d\left((\varphi^{-1})^*\omega\right)\\ &= \psi^*(\psi^{-1})^*\varphi^* d\left((\varphi^{-1})^*\omega\right)\\ &= \varphi^* d\left((\varphi^{-1})^*\omega\right).\end{aligned}$$

So the definition is independent of the coordinate chart.

To show uniqueness, we first need to show that any operator $d$ satisfying (i-iv) is locally determined. Let $\omega_1$ and $\omega_2$ be forms with $\omega_1 = \omega_2$ on an open set $U$, and let $p \in U$. Choose a bump function $\psi$ for $\{p\} \subset U$. We have

$$0 = d\left(\psi(\omega_1 - \omega_2)\right) = d\psi \wedge (\omega_1 - \omega_2) + \psi d(\omega_1 - \omega_2).$$

Evaluating at $p$, we get $d\omega_1(p) = d\omega_2(p)$. Since $p \in U$ was arbitrary, $d\omega_1 = d\omega_2$ on $U$.

Now, by extending $\omega_I$ and $x^i$ to global functions (as in the proof of Lemma 35.4) and using (i-iv), we can see that $d$ must satisfy

$$d\omega = d\left(\sum_I{}' \omega_I dx^I\right) = \sum_I{}' d\omega_I \wedge dx^I,$$

so agrees with the exterior derivative already defined. □

**Proposition 38.3.** $F^* d\omega = dF^*\omega$

*Proof.* This follows from Proposition 38.1$d$ and a similar manipulation to the previous proof (exercise). □

**Definition 38.4.** $\omega \in \Omega^k(M)$ is **closed** if $d\omega = 0$. $\omega$ is **exact** if $\omega = d\eta$ for $\eta \in \Omega^{k-1}(M)$. Notice that exact implies closed, but we will see below that the converse is not always true.

**Example 38.5** (Lee, Example 14.27)**.** Let $M = \mathbb{R}^3$. The standard volume form is $dV = dx \wedge dy \wedge dz$. We have the following standard isomorphisms:

$$\begin{aligned}\flat : \mathscr{X}(\mathbb{R}^3) &\to \Omega^1(\mathbb{R}^3)\\ X^i \frac{\partial}{\partial x^i} &\mapsto \sum_i X^i dx^i\\ \beta : \mathscr{X}(\mathbb{R}^3) &\to \Omega^2(\mathbb{R}^3)\\ X &\mapsto \iota_X dV\\ * : C^\infty(\mathbb{R}^3) &\to \Omega^3(\mathbb{R}^3)\\ f &\mapsto f\, dV.\end{aligned}$$

One can check (exercise) that the three standard operators *grad, curl,* and *div* are the ones that make the following diagram commute:

$$\begin{array}{ccccccc} C^\infty(M) & \xrightarrow{\text{grad}} & \mathscr{X}(M) & \xrightarrow{\text{curl}} & \mathscr{X}(M) & \xrightarrow{\text{div}} & C^\infty(M)\\ \downarrow{\scriptstyle \text{Id}} & & \downarrow{\scriptstyle \flat} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle *}\\ \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(M). \end{array}$$

As a consequence, we get the vector calculus identities $\mathrm{curl} \circ \mathrm{grad} = 0$ and $\mathrm{div} \circ \mathrm{curl} = 0$.

38.3. **Invariant formula for $d$.** Having described it explicitly in local coordinates and shown that it exists globally, we should also find a global/tensorial formula for the exterior derivative. We begin with the case of a 1-form.

**Proposition 38.6.** *Let $\alpha \in \Omega^1(M)$ be a 1-form. For vector fields $X$ and $Y$, we have*

$$(d\alpha)(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \omega([X,Y]).$$

*Proof.* We may let $\alpha = udv$ without loss of generality (by $\mathbb{R}$-linearity). We then have

$$d\alpha = du \wedge dv,$$

so

$$LHS(X,Y) = d\alpha(X,Y) = X(u)Y(v) - Y(u)X(v).$$

We also have

$$RHS(X,Y) = X(u(Y(v)) - Y(u(X(v)) - u\,[X,Y]\,(v).$$

By the Leibniz rule, these agree. ☐

The general formula is somewhat trickier.

**Theorem 38.7.**

$$\begin{aligned} d\omega(X_1,\dots,X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i-1}X_i(\omega(X_1,\dots,\hat{X}_i,\dots,X_{k+1})) \\ &\quad + \sum_{1\le i<j\le k+1}(-1)^{i+j}\omega([X_i,X_j]\,,X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+1}). \end{aligned}$$

*Proof.* We will give a much shorter proof than the one in Lee, based on a mild abuse of notation. Working in a coordinate chart, we write

$$X(Y) = X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j}.$$

The fact that this expression is coordinate-dependent is immaterial, since the final answer will be coordinate-independent.

We write $\omega = \omega_I dx^I$, so that $d\omega = d\omega_I \wedge dx^I$. Notice that

$$\begin{aligned} X(\omega_I)dx^I(Y_1,\dots,Y_k) &= X(\omega(Y_1,\dots,Y_k)) \\ &\quad - \omega(X(Y_1),Y_2,\dots,Y_k) \\ &\quad - \cdots - \\ &\quad - \omega(Y_1,Y_2,\dots,X(Y_k)) \\ &= X\left(\omega(Y_1,\dots,Y_k)\right) - \sum_{j=1}^{k}(-1)^{j-1}\omega\left(X(Y_j),Y_1,\dots,\hat{Y}_j,\dots,Y_k\right). \end{aligned} \tag{38.1}$$

Now, applying a formula similar to (37.2), followed by (38.1), we may write

$$d\omega(X_1,\dots,X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i-1}X_i(\omega_I)dx^I\left(X_1,\dots,\hat{X}_i,\dots X_{k+1}\right)$$

$$= \sum_{i=1}^{k+1}(-1)^{i-1}\left(\begin{array}{r} X_i\left(\omega(X_1,\dots,\hat{X}_i,\dots,X_{k+1})\right) \\ -\sum_{j=1}^{i-1}(-1)^{j-1}\omega\left(X_i(X_j),\dots,\hat{X}_j,\dots,\hat{X}_i,\dots,X_{k+1}\right) \\ +\sum_{j=i+1}^{k+1}(-1)^{j-1}\omega\left(X_i(X_j),\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+1}\right)\end{array}\right).$$

Here, the sign changes after $\hat{X}_j$ passes $\hat{X}_i$ because of the omitted index. The desired formula now follows by relabeling $j$ and $i$ in the second sum and combining $X_i(X_j) - X_j(X_i) = [X_i, X_j]$. □

## 39. Cartan's magic formula (Fri 11/22)

**Lemma 39.1.** $\mathscr{L}_V(\omega\wedge\eta) = \mathscr{L}_V\omega\wedge\eta + \omega\wedge\mathscr{L}_V\eta$.

*Proof.* This follows from the same property of the tensor product, Proposition 36.3. □

**Theorem 39.2** (Cartan's magic formula)**.** $\mathscr{L}_V\omega = (d\circ\iota_V + \iota_V\circ d)\,\omega$.

*Proof.* For the case of a 0-form, we have $\mathscr{L}_V f = V(f) = V \lrcorner\, df$, which is the desired formula (since $V\lrcorner\, f = 0$).

Assume that the formula has been proven up to $k-1$-forms. Assume without loss that $\omega = df\wedge\beta$, which can be achieved by absorbing the "coefficient out front" into $\beta$. Then the LHS of the formula is

$$\begin{aligned}\mathscr{L}_V\omega &= \mathscr{L}_V df\wedge\beta + df\wedge\mathscr{L}_V\beta \\ & dV(f)\wedge\beta + df\wedge\left(V\lrcorner\, d\beta + d(V\lrcorner\,\beta)\right).\end{aligned}$$

The RHS is

$$\begin{aligned}V\lrcorner\, d(df\wedge\beta) + d(V\lrcorner\,(df\wedge\beta)) &= V\lrcorner\,(-df\wedge d\beta) + d\left(V(f)\beta - df\wedge(V\lrcorner\,\beta)\right) \\ &= -V(f)d\beta + df\wedge(V\lrcorner\, d\beta) + d(V(f))\wedge\beta \\ &\quad + V(f)d\beta + df\wedge d(V\lrcorner\,\beta).\end{aligned}$$

Cancelling the $V(f)d\beta$ terms on the expression for the RHS, we see the same three terms as in the expression for the LHS. □

**Remark 39.3.** A slightly longer but perhaps more satisfying proof is based on the observation that both the LHS and RHS are derivations on $\Omega^*(M)$, in the sense of the Lemma, which agree on 0-forms and on exact 1-forms.

**Corollary 39.4.** $\mathscr{L}_V d\omega = d\mathscr{L}_V\omega$.

*Proof.* By Cartan's formula, the LHS is equal to $d(V\lrcorner\, d\omega)$, as is the RHS. □

# Part 8. De Rham cohomology

## 40. The de Rham complex (Fri 11/22-Mon 11/25)

On any smooth manifold $M$, we have constructed the **de Rham complex**

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0.$$

This a sequence of infinite-dimensional real vector spaces (if $n \geq 1$) in which the composition $d \circ d = 0$. As for any complex, we may form the cohomology groups

$$H^k_{dR}(M) := \frac{\ker d : \Omega^k(M) \to \Omega^{k+1}(M)}{\operatorname{im} d : \Omega^{k-1}(M) \to \Omega^k(M)}.$$

These are called the **de Rham cohomology groups** of $M$.

Since $d$ commutes with the pullback operation, we can in fact say that $H^k_{dR}(\cdot)$ is a *contravariant functor* from the category of smooth manifolds and maps to the category of $\mathbb{R}$-vector spaces. This just means that a smooth map $F : M \to N$ induces a linear pullback map $F^* : H^k_{dR}(N) \to H^k_{dR}(M)$, and that these maps are compatible with composition $(F \circ G)^* = G^* \circ F^*$ and respect the identity $\mathbf{1}^* = \mathbf{1}$.

In fact, the deRham groups have a much stronger invariance property.

**Theorem 40.1.** *Suppose that $F, G : M \to N$ are smooth maps which are (smoothly) homotopic, i.e., there exists a smooth map $H_t : M \times [0,1] \to N$ such that $H_0 = F$ and $H_1 = G$. Then*

$$F^* = G^* : H^k_{dR}(N) \to H^k_{dR}(M).$$

*Proof.* Let $\omega$ be a closed $k$-form on $N$. Put $\tilde{\omega} = H^*\omega \in \Omega^k(M \times [0,1])$, which is again closed. Let $i_t : M \to M \times \{t\} \subset M \times [0,1]$ denote the inclusion map of a time-slice. Write $\omega_t = i_t^*\tilde{\omega} = H_t^*\omega$.

Define a $k-1$-form on $M$ by

$$\eta := \int_0^1 i_t^* \left( \frac{\partial}{\partial t} \lrcorner \tilde{\omega} \right) dt.$$

We calculate

$$\begin{aligned} d\eta &= \int_0^1 d\left( i_t^* \left( \frac{\partial}{\partial t} \lrcorner \tilde{\omega} \right) \right) dt \\ &= \int_0^1 i_t^* d\left( \left( \frac{\partial}{\partial t} \lrcorner \tilde{\omega} \right) \right) dt, \end{aligned}$$

since pullback commutes with $d$. Since $\omega$ is closed, we have $\frac{\partial}{\partial t} \lrcorner d\tilde{\omega} = 0$. By Cartan's formula, we obtain

$$\begin{aligned} d\eta &= \int_0^1 i_t^* \mathscr{L}_{\frac{\partial}{\partial t}} \tilde{\omega}\, dt \\ &= \int_0^1 \frac{d}{dt}\omega_t\, dt \\ &= \omega_1 - \omega_0 \\ &= G^*\omega - F^*\omega. \end{aligned}$$

Taking de Rham classes of both sides, we get

$$[G^*\omega] = [F^*\omega] + [d\eta] = [F^*\omega],$$

as claimed. □

**Remark 40.2.** By the Whitney approximation theorem, we can remove "(smooth)" from the statement: two smooth maps are homotopic (via a continuous map) if and only if they are homotopic via a smooth map. Note that the only homotopies we plan to consider will be smooth or at least piecewise smooth.

Recall that a continuous map $f : X \to Y$ between topological spaces is called a *homotopy equivalence* if it has a homotopy inverse, i.e., a map $g : Y \to X$ such that $g \circ f \simeq \mathbf{1}_X$ and $f \circ g \simeq \mathbf{1}_Y$.

**Corollary 40.3.** *If $M$ and $N$ are (smoothly) homotopy equivalent, then $H^k_{dR}(M) \cong H^k_{dR}(N)$ for all $k$.*

*Proof.* If there exist smooth maps $F$ and $G$ as in the definition just given, then $F^* \circ G^* = \mathbf{1}^* = \mathbf{1}$, and similarly for $G^* \circ F^*$, so the induced maps on cohomology are isomorphisms. (One can also approximate a continuous homotopy equivalence by a smooth one to obtain the same result.) □

**Corollary 40.4.** *If $M$ is (smoothly) contractible, then $H^k_{dR}(M) = 0$ for all $k \geq 1$.*

*Note:* One always has $H^0_{dR}(M) = \Pi_{\pi_0(M)}\mathbb{R}$, where the direct product runs over all the connected components of $M$.

**Corollary 40.5** (Poincaré Lemma)**.** *Let $U \subset \mathbb{R}^n$ be a star-shaped open set. We have $H^k_{dR}(U) = 0$ for $k \geq 1$. For example, the de Rham groups of $\mathbb{R}^n, \mathbb{H}^n$, and $B^n$ all vanish for $k \geq 1$. In particular, every closed form is exact!*

**Example 40.6.** Let $M = S^1$. Consider the closed 1-form "$d\theta$," which is in quotation marks because $\theta$ is only well-defined on $S^1$ modulo $2\pi$. This form is not exact, because $\int_0^{2\pi} d\theta = 2\pi$ whereas for any function $f$ on $S^1$ we must have

$$\int_0^{2\pi} df = \int_0^{2\pi} f'(\theta) d\theta = f(2\pi) - f(0) = 0.$$

So the class $[d\theta]$ is nonzero in $H^1_{dR}(S^1)$.

*Claim.* $H^1_{dR}(S^1) = \langle [d\theta] \rangle \cong \mathbb{R}$.

*Proof of claim.* Let $\alpha$ be a closed 1-form. Since $d\theta \neq 0$ pointwise, we have $\alpha = f(\theta)d\theta$ for some function $f(\theta)$. Let

$$c = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta$$

and

$$g(t) = \int_0^t (f(t) - c)\, dt.$$

Then

$$g(2\pi) = \int_0^{2\pi} f(\theta)\, d\theta - 2\pi c = 0 = g(0),$$

so that $g$ is continuous (indeed, smooth) on $S^1$. By the fundamental theorem of calculus, we have

$$dg = \frac{\partial g}{\partial \theta} d\theta = f(\theta)d\theta - c d\theta = \alpha - c d\theta.$$

Taking de Rham classes of both sides, we get

$$[\alpha] = c[d\theta].$$

◇

We conclude that

$$H^k_{dR}(S^1) = \begin{cases} \mathbb{R} & k = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

**Example 40.7.** Let $M = \mathbb{R}^2 \setminus \{0\}$. This is (smoothly) homotopy equivalent to $S^1$, as follows: let $\iota : S^1 \to \mathbb{R}^2 \setminus \{0\}$ be the inclusion and $\pi : \mathbb{R}^2 \setminus \{0\} \to S^1$ the map sending $x \mapsto \frac{x}{|x|}$. The composition $\pi \circ \iota$ is equal to the identity, while we can make a smooth homotopy

$$H_t(x) = tx + (1 - t)\frac{x}{|x|}$$

with $H_0 = \iota \circ \pi$ and $H_1 = \mathbf{1}$. We conclude that

$$H^k_{dR}(\mathbb{R}^2 \setminus \{0\}) \cong H^k_{dR}(S^1) = \begin{cases} \mathbb{R} & k = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is interesting because the manifolds in question do not even have the same dimension.

The idea behind these examples generalizes as follows.

**Theorem 40.8** (De Rham Theorem in 1d)**.** *Let $M$ be a connected smooth manifold and $p \in M$. The "de Rham map"*

$$\Phi : H^1_{dR}(M) \to \operatorname{Hom}(\pi_1(M, p), \mathbb{R})$$
$$[\alpha] \mapsto \Phi[\alpha][\gamma] = \int_\gamma \alpha$$

*is well-defined and injective (indeed, an isomorphism). Here, we define*

$$\int_\gamma \alpha := \int_0^{2\pi} g(\theta)d\theta, \text{ where } \gamma^*\alpha = g(\theta)d\theta.$$

*Proof.* We will prove the well-definedness and leave the injectivity as an exercise (sketched in class, on HW13). The fact that the map is an isomorphism is deeper and we will not quite have time to prove it in 761.

To show that the map is well-defined, we have to check that it is independent both of the representative of $[\alpha] \in H^1_{dR}(M)$ and of $[\gamma] \in \pi_1(M, p)$. If $\alpha_1 = \alpha_2 + df$, for $f \in C^\infty(M)$, then $\gamma^*\alpha_1 = \gamma^*\alpha_2 + d(f \circ \gamma)$, so their integrals over $S^1$ agree. If $\gamma_1 \simeq \gamma_2 : S^1 \to M$ are homotopic paths, then by Theorem 40.1, we have $\gamma_1^*\alpha = \gamma_2^*\alpha + dg$, so the integrals over $S^1$ again agree. □

**Corollary 40.9.** *Suppose that $M$ has finite fundamental group. Then any closed 1-form on $M$ is exact.*

*Proof.* Since there are no nonzero homomorphisms from a finite group to $\mathbb{R}$, the image of any closed form under the de Rham map is zero. By injectivity, it is exact. □

We will define the de Rham map in all dimensions after Thanksgiving.

## 41. The Mayer-Vietoris sequence (Wed 11/27)

Today we will describe a general method to compute the de Rham groups of a manifold, $M$, by breaking it up into smaller pieces.

Suppose that $M = U \cup V$, where $U$ and $V$ are both open. We have a diagram of smooth maps

$$\begin{array}{ccccc} & & U & & \\ & \overset{i}{\nearrow} & & \overset{k}{\searrow} & \\ U \cap V & & & & M, \\ & \underset{j}{\searrow} & & \underset{\ell}{\nearrow} & \\ & & V & & \end{array}$$

which are just the relevant restriction maps. This gives us a diagram of vector spaces

$$\begin{array}{ccccc} & & \Omega^p(U) & & \\ & \overset{i^*}{\swarrow} & & \overset{k^*}{\nwarrow} & \\ \Omega^p(U \cap V) & & & & \Omega^p(M). \\ & \underset{j^*}{\nwarrow} & & \underset{\ell^*}{\swarrow} & \\ & & \Omega^p(V) & & \end{array}$$

From this, we can form the sequence

$$0 \to \Omega^p(M) \xrightarrow{k^* \oplus \ell^*} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{i^* - j^*} \Omega^p(U \cap V) \to 0. \tag{41.1}$$

This sequence is exact. For, the second map is clearly injective, since a form in $M$ vanishes if and only if its restrictions to $U$ and $V$ both vanish. The kernel of the third map is equal to the image of the second map because $(\eta, \mu) \in \Omega^p(U) \oplus \Omega^p(V)$ comes from a form $\omega \in \Omega^p(M)$ if and only if $\eta_{U \cap V} = \mu|_{U \cap V}$. To show surjectivity of the last map, let $\{\rho, \psi\}$ be a partition of unity subordinate to $\{U, V\}$. Given $\lambda \in \Omega^p(U \cap V)$, put

$$\eta = \begin{cases} \psi\lambda & \text{on } U \cap V \\ 0 & \text{on } U \setminus V \end{cases}$$

and

$$\mu = \begin{cases} -\rho\lambda & \text{on } U \cap V \\ 0 & \text{on } V \setminus U. \end{cases}$$

These are smooth forms on $U$ and $V$, respectively, and we have

$$(i^* - j^*)(\eta, \mu) = \psi\lambda + \rho\lambda = (\psi + \rho)\lambda = \lambda,$$

as required.

We have shown that (41.1) is a short exact sequence of groups; in fact, because these restriction maps all commute with $d$, we can put the full de Rham complexes together into a **short exact sequence of complexes**

(41.2)
$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{p}(M) & \xrightarrow{k^* \oplus \ell^*} & \Omega^{p}(U) \oplus \Omega^{p}(V) & \xrightarrow{i^* - j^*} & \Omega^{p}(U \cap V) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d \oplus d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & \Omega^{p+1}(M) & \xrightarrow{k^* \oplus \ell^*} & \Omega^{p+1}(U) \oplus \Omega^{p}(V) & \xrightarrow{i^* - j^*} & \Omega^{p+1}(U \cap V) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where the whole diagram commutes. The **snake lemma**, which is a purely algebraic fact, says that any short exact sequence of complexes of abelian groups (or elements of any abelian category) gives rise to a **long exact sequence of cohomology groups**:

(41.3) $$\cdots \to H^{p-1}(U \cap V) \xrightarrow{\delta} H^{p}(M) \xrightarrow{k^* \oplus \ell^*} H^{p}(U) \oplus H^{p}(V) \xrightarrow{i^* - j^*} H^{p}(U \cap V) \xrightarrow{\delta} H^{p+1}(M) \to \cdots.$$

Here, $\delta$ is the so-called **connecting homomorphism**, which is gotten by tracing through the diagram as explained in class. In this situation, we have explicitly

$$\delta([\lambda]) = \left[ \begin{cases} d(\psi\lambda) \text{ on } U \\ -d(\rho\lambda) \text{ on } V \end{cases} \right].$$

This form is closed, since it is locally exact, so defines a cohomology class; but it may not be globally exact, i.e. $\delta([\lambda])$ may be nonzero in $H^{p+1}(M)$. You are encouraged to think through the diagram chase involved in proving the snake lemma, if you have not done so recently.

We can summarize this discussion as follows.

**Theorem 41.1.** *For any decomposition $M = U \cup V$, the de Rham cohomology groups form a long exact sequence as in (41.3), called the* **Mayer-Vietoris sequence**.

**Theorem 41.2.** *For $n \geq 1$, we have* $H^p_{dR}(S^n) = \begin{cases} \mathbb{R} & p = 0, n \\ 0 & \textit{otherwise.} \end{cases}$

*Proof.* We proceed by induction on $n$. The case $n = 1$ was proved in Example 40.6. Let $n \geq 2$ and suppose the result has been proved up to $n - 1$. We take $U$ and $V$ to be the standard stereographic charts, both diffeomorphic to $\mathbb{R}^n$, so that their de Rham groups vanish except in degree zero. Then $U \cap V$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$, which deformation-retracts onto $S^{n-1}$, so by induction we have

$$H^p_{dR}(U \cap V) = \begin{cases} \mathbb{R} & p = 0, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $S^n$ is connected, we know that $H^0_{dR}(S^n) = \mathbb{R}$. The first part of the Mayer-Vietoris sequence is

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to H^1_{dR}(S^n) \to 0 \to \cdots,$$

with the third map surjective, so that $H^1_{dR}(S^n) = 0$. There later parts of the sequence look like

$$0 \to H^{p-1}_{dR}(S^{n-1}) \to H^p_{dR}(S^n) \to 0,$$

where $p \geq 2$. This gives the desired result. □

**Corollary 41.3.** *For $n \geq 2$, we have* $H^p_{dR}(\mathbb{R}^n \setminus \{0\}) \simeq \begin{cases} \mathbb{R} & n = 0, n-1 \\ 0 & \text{otherwise.} \end{cases}$

*Proof.* We have a (smooth) deformation retraction $\mathbb{R}^n \setminus \{0\} \to S^{n-1}$, so the result follows from the previous theorem. □

**Corollary 41.4.** *Let $U \subset \mathbb{R}^n$, $n \geq 2$, be a nonempty open set. For any $x \in U$, we have*

$$H^{n-1}_{dR}(U \setminus \{x\}) \neq 0.$$

*Proof.* Choose a small sphere $S \cong S^{n-1}$ centered at $x$ and contained in $U$. Let $\pi : U \setminus \{x\} \to S$ be the restriction of the usual retraction from $\mathbb{R}^n \setminus \{x\}$ to $S$, and $\iota : S \to U$ the inclusion map. We have $\pi \circ \iota = \mathbf{1}_S$. Applying the de Rham functor to the sequence of smooth maps

$$S \xrightarrow{\iota} U \setminus \{x\} \xrightarrow{\pi} S,$$

we obtain a sequence

$$\mathbb{R} \cong H^{n-1}_{dR}(S) \leftarrow H^{n-1}_{dR}(U \setminus \{x\}) \leftarrow H^{n-1}_{dR}(S) \cong \mathbb{R}.$$

Since the composition must be the identity, the middle group must be nonzero. □

**Remark 41.5.** This proof can be compared with that of Proposition 34.1, where we could have used de Rham cohomology in place of singular homology.

**Corollary 41.6** (Topological invariance of dimension)**.** *Suppose that $M$ and $N$ are topological manifolds of dimension $m$ and $n$, respectively. If $M$ and $N$ are homeomorphic then $m = n$.*

*Proof.* Assume for contradiction that $m > n$. Let $x \in M$ be arbitrary. Choose $V \ni x$ with $V$ homeomorphic to $\mathbb{R}^m$. By assumption, there also exists $U \subset V$ with $U \ni x$ such that $U$ is homeomorphic to $\mathbb{R}^n$.

First give $U$ the smooth structure coming from the homeomorphism to $\mathbb{R}^m$. Since $m-1 \geq n$, we have $H^{m-1}_{dR}(U \setminus \{x\}) = H^{m-1}_{dR}(\mathbb{R}^n \setminus \{0\}) = 0$ by the first corollary above. On the other hand, we can give $V$ the smooth structure coming from the homeomorphism with $\mathbb{R}^n$, and let $\tilde{U}$ be the induced smooth structure on $U$ from the inclusion $U \subset V$. By the last corollary, we have $H^{m-1}_{dR}(\tilde{U} \setminus \{x\}) \neq 0$. But $U$ and $\tilde{U}$ are both smooth manifolds, and they are homeomorphic. By the Whitney approximation theorem (skipped, see Lee for statement and proof), there exists a pair of smooth maps giving a smooth homotopy equivalence between $U$ and $\tilde{U}$. The above calculation violates the invariance of the de Rham groups under smooth homotopy equivalence.[15] □

[15]Indeed, by this argument, they are invariant under general homotopy equivalence.

# Part 9. Integration of differential forms

## 42. Motivation (Mon 12/2)

The moral of Example 40.6-Theorem 40.8 was that de Rham cohomology is intimately related to integration; we need to learn how to integrate on manifolds.

Let us begin with some motivating discussion. We saw above that for a path $\gamma : [0,1] \to M$, we can form the integral

$$\int_\gamma \alpha := \int_0^1 \gamma^* \alpha = \int_0^1 \alpha(\gamma'(t)) dt.$$

Let us point out (again) that this object is actually invariant under reparametrization of $\gamma$. Let $u : [0,1] \to [0,1]$ be an increasing diffeomorphism and $\tilde{\gamma}(t) = \gamma(u(t))$. By the change-of-variable formula from beginning calculus, we have

$$\int_{\tilde{\gamma}} \alpha = \int_0^1 \alpha(\tilde{\gamma}'(t))\, dt = \int_0^1 \alpha(\gamma'(u(t))u'(t))\, dt = \int_0^1 \alpha(\gamma'(u(t)))u'(t) dt = \int_0^1 \alpha(\gamma'(u))\, du.$$

Therefore the integral is in fact independent of the parametrization of the path from zero to one. This suggests that the quantity $\int_\gamma \alpha$ represents something like "displacement" along the path $\gamma$, i.e., a signed measure of the length of $\gamma$. More strikingly, if the form $\alpha$ happens to be *closed*, then the integral is also invariant under *homotopies* of the path that fix the endpoints, by the proof of Theorem 40.8; this can also be seen using the classical Green's theorem. (In the case that the codomain manifold $M$ has dimension one, all 1-forms are closed, which gives another way to explain the invariance under general reparametrizations of the domain.)

Let us also discuss the case of top forms ($k = n$) geometrically. We have seen above that

$$dx^1 \wedge \cdots \wedge dx^n(v_1, \ldots, v_n) = \begin{vmatrix} v_1 & \cdots & v_n \end{vmatrix}.$$

It is usually shown in a linear algebra class that this is equal to $\pm$ the volume of the parallelipiped spanned by $v_1, \ldots, v_n$, where $\pm$ is determined by whether or not $\{v_1, \ldots, v_n\}$ corresponds to the standard orientation of $\mathbb{R}^n$. (This can be proved by observing that row operations correspond to surgeries on the parallelipiped of the type we know from the 2D case, which do not change the volume, and by which one can reduce to the case of an $n$-dimensional rectangle.) So differential forms of general dimension measure (signed) volume, which should have something to do with integration.

This discussion suggests that differential forms are indeed the right things to try to integrate over manifolds, although we have already seen that there will be a subtlety involving signs.

## 43. Integration on $\mathbb{R}^n$ (Mon 12/2)

We begin by defining the integral of an $n$-form on $\mathbb{R}^n$. Recall that any top form is proportional to $dx^1 \wedge \cdots \wedge dx^n$ at each point.

**Definition 43.1.** Let $D \subset \mathbb{R}^n$ be an open domain of integration, i.e. an open set whose boundary has measure zero. Given a top form $\omega = f dx^1 \wedge \cdots \wedge dx^n$ for which $f$ extends continuously to $\bar{D}$ with compact support, we define

$$\int_D \omega := \int_D f dx^1 \cdots dx^n,$$

this being the Riemann integral.

Here is our first generalization of the Fundamental Theorem of Calculus to higher dimensions.

**Theorem 43.2** (Stokes's Theorem, first version)**.** *For $\omega \in \Omega_c^{n-1}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} d\omega = 0.$$

*Proof.* We may write

$$\omega = \omega_i dx^i \wedge \cdots \wedge \hat{dx}^i \wedge \cdots \wedge dx^n.$$

Then

$$d\omega = \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n.$$

The integral of each term vanishes:

$$\int_{\mathbb{R}^n} \frac{\partial \omega_i}{\partial x^i} dx^1 \cdots dx^n = \lim_{R \to \infty} \int_{-R}^R \cdots \left[ \int_{-R}^R \frac{\partial \omega_i}{\partial x^i} dx^i \right] dx^1 \cdots \hat{dx}^i \cdots dx^n$$

by Fubini's theorem. Since $\omega$ has compact support, for $R$ sufficiently large, we have

$$\int_{-R}^R \frac{\partial \omega_i}{\partial x^i} dx^i = \omega_i \big|_{x^i = R} - \omega_i \big|_{x^i = -R} = 0.$$

□

**Remark 43.3.** Later, we will obtain a version of Stokes's theorem for a domain $D \subset \mathbb{R}^n$ with smooth boundary that you may have seen before. One can actually get this directly from the previous theorem by inserting a cutoff $\varphi$:

$$0 = \int_{\mathbb{R}^n} d(\varphi \omega) = \int \varphi d\omega + \int d\varphi \wedge \omega.$$

If one lets $\varphi$ approach the characteristic function of $D$, then the first term approaches $\int_D d\omega$; while if $d\varphi$ also approaches a "$\delta$-function" concentrated on the boundary of $D$, then the second term approaches $\pm \int_{\partial D} \omega$. We will not take this approach, but will instead learn first how to integrate over general manifolds (with boundary) in order to make sense of the boundary term.

We now come to the key property that will allow us to define integrals over manifolds.

**Theorem 43.4** (Change-of-variables)**.** *Let $D, E \subset \mathbb{R}^n$ be open sets and $\varphi : D \to E$ a diffeomorphism. If $\omega$ is integrable on $E$ and $\varphi$ is orientation-preserving, then*

$$\int_D \varphi^* \omega = \int_E \omega.$$

*If $\varphi$ is orientation-reversing, we have*

$$\int_D \varphi^* \omega = -\int_E \omega.$$

*First proof.* Letting $\omega = f(x)dx^1 \wedge \cdots \wedge dx^n$, Proposition 37.7(c) gives

$$\varphi^*\omega = f \circ \varphi \det \varphi dx^1 \wedge \cdots \wedge dx^n = f \circ \varphi |\det \varphi| dx^1 \wedge \cdots \wedge dx^n$$

since $\varphi$ is orientation-preserving. This gives us

$$\int_D \varphi^*\omega = \int_D (f \circ \varphi) |\det \varphi| dx^1 \cdots dx^n.$$

By the ordinary change-of-variable formula from multivariable calculus (which is a big pain to prove), the latter is equal to $\int_E \omega$. □

*Second proof.* We give another proof under the assumption that $\varphi$ extends to an orientation-preserving diffeomorphism of $\mathbb{R}^n$ such that $\varphi \equiv \mathbf{1}$ outside a large compact set, and $\omega \in \Omega_c^n(\mathbb{R}^n)$. In this case, we can define a smooth map

$$\varphi_t = (1-t)\varphi + t\mathbf{1} : \mathbb{R}^n \to \mathbb{R}^n.$$

Let $X_t = \frac{d\varphi_t}{dt}$, which is compactly supported. By definition of the Lie derivative and Cartan's formula, we have

$$\frac{d}{dt}\varphi_t^*\omega = \mathscr{L}_{X_t}\varphi_t^*\omega = d(X_t \lrcorner \varphi_t^*\omega),$$

since $d\omega = 0$ for a top form. Integrating from zero to one (as in the proof of Theorem 40.1), we get

$$\omega - \varphi^*\omega = \varphi_1^*\omega - \varphi_0^*\omega = d\left(\int_0^1 X_t \lrcorner \varphi_t^*\omega\right).$$

Integrating both sides and applying Theorem 43.2, we obtain the result. □

**Remark 43.5.** With regard to the second proof, here are two things to think about: how can it be extended to cover the general case? And, what goes wrong with the argument when $\varphi$ is not orientation-preserving?

## 44. Orientation of manifolds (Mon 12/2-Wed 12/4)

We have seen above that integration is sensitive to orientation. We have already discussed orientations of general vector bundles; below, we will simply define a manifold to be **orientable** if and only if its tangent bundle is orientable. Before discussing this in detail, we include a piece that could already have appeared in §33.

44.1. **Double cover corresponding to a real line bundle.** Let $L \to M$ be a real line bundle. Since each fiber $L_x$ is isomorphic to $\mathbb{R}$, $L_x \setminus \{0\}$ consists of exactly two components $\{[v], [-v]\}$, where $v \in L_x$ is any nonzero vector and $[v]$ denotes the orbit of $v$ under multiplication by $\mathbb{R}_+$. We can define a manifold $M_L$, which comes with a 2-to-1 covering map $\pi : M_L \to M$, as follows. As a set, $M_L$ is equal to the set of all points

$$(x, [v]),$$

where $v \in L_x$. To give $M_L$ the structure of a manifold, one can post-compose the charts of $M$ with arbitrary *nonvanishing* local sections of $L \to M$ (which exist because $L$ is locally trivial). The transition functions of $M_L$ are then identical to those of $M$, and by the smooth manifold chart lemma, $M_L$ is a manifold. It is an exercise on HW13 to check the details of this construction.

**Lemma 44.1.** *Suppose that $M$ is connected. Then $L \to M$ is trivial $\iff$ $M_L$ is disconnected.*

*Proof.* (Sketch) If $L$ is trivial then it has a nonvanishing global section $s$. Then $-s$ is also a nonvanishing global section, and we have $s(M) \sqcup -s(M) = M_L$. For the other direction, if $M_L$ is disconnected, then it has exactly two connected components. A global nonvanishing section of $M_L \to M$ can be defined by choosing one of the components and mapping each point $x \in M$ to the point in $\pi^{-1}(x)$ that lies in that component. Using a partition of unity, one can lift this section of $\pi$ to a global section of $L$. It is again an exercise on HW13 to check the details. □

**Example 44.2.** If $L$ is the Möbius bundle over $S^1$, the orientation double cover $M_L$ is again $S^1$. This is connected, i.e., the Möbius strip is nontrivial.

44.2. **The orientation double cover.** We now apply the previous construction to the case $L = T^*M$ : let

$$\hat{M} := M_{\Lambda^n T^* M},$$

and $\hat{\pi} : \hat{M} \to M$ be the corresponding 2-to-1 projection map. This is called the **orientation** (or *orientable*) **double cover** of $M$.

**Definition/Lemma 44.3.** *A connected smooth manifold $M$ is called* **orientable** *if the following equivalent conditions are true.*

1. *$TM$ is an orientable vector bundle over $M$ (see Definition 33.1).*
2. *$T^*M$ is an orientable vector bundle over $M$*
3. *$\hat{M}$ is disconnected*
4. *There exists an atlas $\{(U_a, \varphi_a)\}$ such that the transition maps are all orientation-preserving*
5. *For each $[\gamma] \in \pi_1(M,p)$, the lift $\hat{\gamma} : [0,1] \to \hat{M}$ is closed, i.e. $\hat{\gamma}(0) = \hat{\gamma}(1)$.*

*Proof.* Think through this as an exercise. The proof that (4) is equivalent to (1-2) is very similar to that of Proposition 33.5. □

**Example 44.4.** Consider the total space of the Möbius bundle, per Example 1.9. Letting $\gamma : [0,1] \to S^1 \times \{0\}$ be the zero section, it is easy to write down a frame over $\gamma$, which corresponds to a lift $\hat{\gamma}$ to the orientation cover, so that the frames at $t = 0$ and at $t = 1$ correspond to opposite orientations. This means that $\hat{\gamma}$ is not a closed loop, so the Möbius strip is not orientable.

We will also need the following criterion below.

**Corollary 44.5** (Lee, Theorem 15.36)**.** *Suppose that there exists a connected, orientable, normal*[16] *covering space* $\pi : N \to M$. *Then* $M$ *is orientable if and only if* $\mathrm{Aut}_\pi(M)$, *the group of diffeomorphisms* $\alpha : N \to N$ *such that* $\pi \circ \alpha = \pi$, *consists only of orientation-preserving diffeomorphisms.*

*Proof.* For a normal covering space, the elements of $\mathrm{Aut}_\pi(M)$ are induced by lifting loops in $M$. A little bit of thought using (5) of the previous Definition/Lemma gives the result. (See Lee for an alternative argument.) ☐

44.3. **Induced orientation.** Given a submanifold $S \subset M$ and a section $N \in \Gamma(TM|_S)$ such that $N_x \notin T_xS$ for all $x \in S$, we may define the **induced orientation** on $S$ as follows:

$$\{v_1, \dots, v_{n-1}\} \subset T_xS \text{ is oriented } \iff \{N_x, v_1, \dots, v_{n-1}\} \subset T_xM \text{ is oriented.}$$

**Example 44.6.** The **standard orientation** on $S^n \subset \mathbb{R}^{n+1}$ is, by definition, the orientation induced by the outward-pointing normal $N_x = x^i \frac{\partial}{\partial x^i}$. For example, at the north pole $p \in S^2$, we have $N_p = \frac{\partial}{\partial z}$. Since $\left[\{\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}\right] = \left[\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}\right]$, the frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is oriented at $p \in S^2$. Since the upper stereographic chart is connected, $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is an oriented frame there; in particular, $\{x, y\}$ is an oriented coordinate system. The lower stereographic chart has to be reversed.

Now consider the projection $\pi : S^n \to \mathbb{RP}^n$. The automorphism group $\mathrm{Aut}_\pi = \langle \pm \mathbf{1} \rangle$. Since the antipodal map preserves the outward normal, it preserves the orientation on $S^n$ if and only if it preserves the orientation on $\mathbb{R}^{n+1}$. This is the case if and only if $n + 1$ is even. By Corollary 44.5, we conclude that

$$\mathbb{RP}^n \text{ is orientable } \iff n \text{ is odd.}$$

We knew this for $n = 1$ since $\mathbb{RP}^1 \cong S^1$, and for $n = 3$ since

$$\mathbb{RP}^3 \cong S^3/\pm 1 \cong \mathrm{SU}(2)/\pm 1 \cong \mathrm{SO}(3)$$

is a Lie group (by §28), hence parallelizable.

## 45. Integration on manifolds (Wed 12/4-Fri 12/6)

**Definition/Lemma 45.1.** *Let* $M$ *be an oriented smooth manifold of dimension* $n$. *Given a compactly-supported top form* $\omega \in \Omega^n_c(M)$, *the* **integral** *of* $\omega$ *over* $M$ *is defined as follows. Let* $\{(U_i, \varphi_i)\}$ *be a locally finite oriented atlas (which exists by Definition/Lemma 44.3.4) and subordinate partition of unity* $\{\rho_i\}$, *and define*

$$\int_M \omega := \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i \omega).$$

[16]By definition, this means that $\mathrm{Aut}_\pi$ acts transitively on fibers. Equivalently, the image of $\pi_1(N)$ in $\pi_1(M)$ is a normal subgroup. In particular, if $N$ is simply connected (i.e. the universal cover), then $\pi$ is normal.

*This is independent of the oriented atlas and partition of unity, and invariant under orientation-preserving diffeomorphisms. I.e., if $F : M \to N$ is a diffeomorphism between two oriented manifolds that preserves orientation, then*

$$\int_M F^*\omega = \int_N \omega$$

*for every $\omega \in \Omega_c^n(N)$.*

*Proof.* Let $\{(\tilde{U}_j, \tilde{\varphi}_j)\}$, be a different cover with the same orientation, and $\{\tilde{\rho}_j\}$ a subordinate partition of unity. We must check that the definitions agree. Notice that since $\omega$ is compactly supported and the covers are locally finite, all of the sums involved are actually finite, so we can manipulate them without worrying. We have

$$\begin{aligned} \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i\omega) &= \sum_i \int_{\varphi_i(U_i)} (\sum_j \tilde{\rho}_j)(\varphi_i^{-1})^*(\rho_i\omega) \\ &= \sum_{i,j} \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\tilde{\rho}_j\rho_i\omega) \\ &= \sum_{i,j} \int_{\varphi_i(U_i\cap\tilde{U}_j)} (\varphi_i^{-1})^*(\tilde{\rho}_j\rho_i\omega). \end{aligned} \tag{45.1}$$

We apply change-of-variables, Theorem 43.4, to the transition function $\varphi_i \circ \tilde{\varphi}_j^{-1} : \tilde{\varphi}_j(\tilde{U}_j \cap U_i) \to \varphi_i(U_i \cap \tilde{U}_j)$, to obtain

$$\begin{aligned} (45.1) &= \sum_{i,j} \int_{\tilde{\varphi}_j(\tilde{U}_j\cap U_i)} (\tilde{\varphi}_j^{-1})^*(\varphi_i)^*(\varphi_i^{-1})^*(\rho_j\rho_i\omega) \\ &= \sum_{i,j} \int_{\tilde{\varphi}_j(\tilde{U}_j))} (\tilde{\varphi}_j^{-1})^*(\rho_i\rho_j\omega) \\ &= \sum_j \int_{\tilde{\varphi}_j(\tilde{U}_j)} (\tilde{\varphi}_j^{-1})^* \left( (\sum_i \rho_i)\rho_j\omega \right) \\ &= \sum_j \int_{\tilde{\varphi}_j(\tilde{U}_j)} (\tilde{\varphi}_j^{-1})^* (\rho_j\omega). \end{aligned}$$

This completes the proof of independence.

To show diffeomorphism-invariance, choose a system of oriented charts and a partition-of-unity on $N$, and simply pull this back by $F$ to a system on $M$. Theorem 43.4 shows that the sum defining the integral on $N$ is equal term-by-term with the sum defining the integral on $M$. □

**Remark 45.2** (Friday 12/6)**.** Let us point out that this is the only possible definition of the integral which is both additive and diffeomorphism-invariant. For, we have

$$\begin{aligned} \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i\omega) &= \sum_i \int_{U_i} \varphi_i^*(\varphi_i^{-1})^*(\rho_i\omega) \\ &= \sum_i \int_M \rho_i\omega \\ &= \int_M \sum_i \rho_i\omega = \int_M \omega. \end{aligned}$$

**Theorem 45.3** (Stokes's Theorem, 2nd version)**.** *For $\omega \in \Omega_c^{n-1}(M)$, we have*

$$\int_M d\omega = 0.$$

*Proof.* We have

$$\begin{aligned}\int_M d\omega = \int_M d(\sum_i \rho_i \omega) &= \sum_i \int_M d(\rho_i \omega) \\ &= \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* d(\rho_i \omega) \\ &= 0.\end{aligned}$$

Here we have used diffeomorphism-invariance of the integral and the first version, Theorem 43.2. ☐

**Remark 45.4.** This result actually implies the full version of Stokes that we will prove below; see Remark 43.3 above. However, it makes more sense to first introduce manifolds with boundary and define the two sides of the Theorem independently, before proving that they agree.

**Example 45.5** (Friday 12/6)**.** Let $G$ be a Lie group. Choose an orientation for $T_eG$, and let $\omega$ be a left-invariant $n$-form on $G$ which is compatible with the orientation at $e$. Then $\omega$ defines an orientation on $G$ with respect to which $\omega$ is positive at each point. Notice that by left-invariance, $\omega$ is unique up to scaling by a positive constant.

Is $\omega$ also right-invariant? Consider $R_g^*\omega$. We have

$$L_h^* R_g^* \omega = R_g^* L_h^* \omega = R_g^* \omega,$$

so that $R_g^*\omega$ is again left-invariant. We must therefore have

$$R_g^*\omega = \lambda_g \omega$$

for some constant $\lambda_g$.

Suppose now that $G$ is compact, and choose $\omega$ such that $\int_G \omega = 1$. By diffeomorphism invariance of the integral, we have

$$\int_G R_g^*\omega = \int_G \omega = 1.$$

This gives

$$\lambda_g = 1$$

for all $g \in G$, which shows that $\omega$ is in fact also right-invariant.

We have proven that every compact Lie group has a unique bi-invariant probability measure, called the *Haar measure*. One can define the integral of a function by

$$\int_G f = \int_G f\omega.$$

This is a very useful fact in geometry as well as in the representation theory of compact Lie groups. One very interesting theorem to mention is the following. For a Lie group $G$ with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, let $\mathfrak{g}^*$ denote the space of left-invariant 1-forms on $G$. The $k$'th wedge product $\mathfrak{g}^*$ is equal to the space of left-invariant $k$-forms. Left-invariance is preserved by $d$, so these spaces form a complex

$$0 \to \mathbb{R} \overset{d}{\to} \mathfrak{g}^* \overset{d}{\to} \Lambda^2\mathfrak{g}^* \overset{d}{\to} \cdots \overset{d}{\to} \Lambda^n\mathfrak{g}^* \to 0,$$

which is a finite-dimensional sub-complex of the de Rham complex. Its differentials can be computed just from the structure constants of $\mathfrak{g}$. The following can be proved by a careful averaging procedure using the Haar measure.

**Theorem 45.6** (Cartan-Eilenberg)**.** *Suppose that $G$ is compact and connected. The inclusion $\Lambda^*\mathfrak{g}^* \to \Omega^*(G)$ induces isomorphisms on all cohomology groups.*

## 46. Integration on manifolds with boundary (Wed 12/4-Fri 12/6)

**Definition 46.1** (Wednesday 12/4)**.** A **smooth manifold with boundary** is a Hausdorff, second-countable topological space endowed with a smooth atlas consisting of charts whose codomains are open sets in

$$\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$$

in the subspace topology.

The charts of a manifold with boundary fall into two types: those whose codomain is an open set entirely contained in $\mathring{\mathbb{H}}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n > 0\}$, which are the same as ordinary smooth manifold charts, and those which intersect $\mathbb{R}^{n-1} \times \{0\}$ nontrivially. Note that a chart of the first kind can never be diffeomorphic to a chart of the second kind. Also note that for two charts of the second kind, because the transition map is required to be a diffeomorphism, it must preserve $\mathbb{R}^{n-1} \times \{0\}$. The **boundary**

$$\partial M = \{x \in M \mid x \in \varphi^{-1}\left(\mathbb{R}^{n-1} \times \{0\}\right) \text{ for some chart } \varphi\}$$

is therefore a smooth manifold *without boundary* of dimension $n - 1$.

Almost all of what we know for smooth manifolds works for smooth manifolds with boundary:

- A manifold with boundary has a tangent bundle $TM$ which is a rank $n$ smooth bundle over $M$. The tangent bundle of the boundary
$$T\partial M \subset TM|_{\partial M}$$
is a subbundle of the restriction, of rank $n - 1$.

- Partitions of unity still work. Note that the restriction of a partition of unity to $\partial M$ is a partition of unity.

- A vector field $X$ is a section of $\Gamma(TM)$. We say that $X$ is *tangent to the boundary* if $X|_{\partial M}$ is a section of the subbundle $T\partial M \subset TM|_{\partial M}$.

- A vector field $X$ tangent to $\partial M$ generates a flow that preserves $\partial M$.

- Differential forms are the same.
- Integration is the same.

**Definition 46.2.**
- Let $v \in T_xM$ for $x \in \partial M$. We say that $v$ is **outward-pointing** if there exists a smooth path $\gamma : (-\varepsilon, 0] \to M$ with $\gamma(0) = x$ and $\gamma'(0) = v$, and **strictly outward-pointing** if also $v \notin T_x\partial M$. Inward-pointing is the same.
- Suppose that $M$ is oriented. The **Stokes orientation** on $\partial M$ is the induced orientation (in the sense of Definition 44.3) by any strictly outward-pointing vector field $N$ on $\partial M$. In other words, $\{v_1, \dots, v_{n-1}\}$ is positively oriented in $T_x\partial M$ iff $\{N_x, v_1, \dots, v_{n-1}\}$ is positively oriented in $T_xM$.

**Examples 46.3.**
- The **upper-half-space** $\mathbb{H}^n$ is itself a manifold with boundary with a single chart, as is any open subset. The boundary $\partial\mathbb{H}^n$ is equal to $\mathbb{R}^{n-1} \times \{0\}$ but possibly with a different orientation:

$$\left[\left\{-\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\right\}\right] = (-1)^n \left[\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}\right]. \tag{46.1}$$

- The **closed unit ball** $B^n = \{x \in \mathbb{R}^n \mid |x| \le 1\}$ is a manifold with boundary $\partial B^n = S^{n-1}$. By definition, the Stokes orientation agrees with the standard orientation (see §44.3).

**Theorem 46.4** (Stokes's Theorem, 3rd version)**.** *Suppose that $M$ is an oriented manifold with boundary and $\partial M$ is given the Stokes orientation. For any $\omega \in \Omega^n_c(M)$, we have*

$$\int_{\partial M} \omega = \int_M d\omega.$$

*Proof.* We first prove the special case $M = \mathbb{H}^n$; let $\omega \in \Omega^n_c(\mathbb{H}^n)$. As in the proof of Theorem 43.2, we write

$$\omega = \omega_i dx^1 \wedge \dots \wedge \hat{dx^i} \wedge \cdots dx^n$$

and

$$d\omega = \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \dots \wedge dx^n.$$

This gives

$$\int_{\mathbb{H}^n} d\omega = \sum (-1)^{i-1} \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{\partial \omega_i}{\partial x^i} dx^1 \cdots dx^n.$$

By the 1D fundamental theorem of calculus (see the proof of Theorem 43.2), all terms vanish except the last one. So the latter expression is equal to

$$(-1)^{n-1} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \int_0^\infty \frac{\partial \omega_n}{\partial x^n} dx^n dx^1 \cdots dx^{n-1} = (-1)^n \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \omega_n(x^1, \dots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.$$

In view of (46.1), the chart $\{x^1, \dots x^{n-1}\}$ is $(-1)^n$-oriented on $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1} \times \{0\}$. Therefore the last expression is equal to

$$\int_{\partial\mathbb{H}^n} \omega,$$

as claimed.

For the general case, let $\{(U_i, \varphi_i)\}$ be an oriented atlas for $M$ and $\{\rho_i\}$ a subordinate partition of unity. We have

$$\begin{aligned}\int_M d\omega = \sum_i \int_M d(\rho_i \omega) &= \sum_i \int_{\mathbb{H}^n} d\big((\varphi_i^{-1})^* \rho_i \omega\big) \\ &= \sum_i \int_{\partial \mathbb{H}^n} (\varphi_i^{-1})^* \rho_i \omega \\ &= \sum_i \int_{\partial M} \rho_i \omega = \int_{\partial M} \omega,\end{aligned}$$

where we have used diffeomorphism invariance and the first case. □

Here is a simple but interesting corollary.

**Corollary 46.5.** *Let $M$ be a compact, oriented manifold with boundary. There does not exist a (smooth) retraction $r : M \to \partial M$.*

*Proof.* Suppose for contradiction that there exists a smooth retraction $r$. Let $\omega$ be any positively oriented $(n-1)$-form on $\partial M$ (with the Stokes orientation). We have

$$0 < \int_{\partial M} \omega = \int_{\partial M} r^* \omega$$

since $r|_{\partial M} = \mathbf{1}$. Note that $r^*\omega$ is a globally defined form on $M$, which has compact support since $M$ is compact. Stokes's Theorem gives

$$0 < \int_{\partial M} r^* \omega = \int_M d(r^* \omega) = \int_M r^*(d\omega).$$

But at the last step, $d$ is the exterior derivative on $\partial M$, which has dimension $n-1$, so that $d\omega = 0$. We have proven $0 < 0$, a contradiction. □

# Part 10. Cohomology and integration

## 47. Compactly-supported de Rham cohomology and the integration map (Fri 12/6)

We have seen above that integration and exterior differentiation are related via Stokes's Theorem. Two subtleties are the question of orientation and the requirement that the form in question be compactly supported. In order to describe the precise relationship between integration and de Rham cohomology, it is convenient to introduce the following variant.

**Definition 47.1.**

- The de Rham cohomology groups **with compact support**, $H^p_c(M)$, are the cohomology groups of the compactly-supported de Rham complex:

$$0 \to C^\infty_c(M) \xrightarrow{d} \Omega^1_c(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_c(M) \to 0.$$

  Notice that these agree with the ordinary de Rham groups in the case that $M$ is compact.

- Suppose that $M$ is an oriented manifold without boundary. Define the **integration map**

$$I_M : H^n_c(M) \to \mathbb{R}$$
$$[\omega] \mapsto \int_M \omega.$$

  This is well-defined in view of Stokes's Theorem: for $\eta \in \Omega^{n-1}_c(M)$, we have $\int_M d\eta = 0$, so (compact-supportedly) cohomologous forms have the same integral.

**Example 47.2.**

- $M = S^n$. Observe that $\omega \in \Omega^n(S^n)$ is exact if and only if $I_{S^n}(\omega) = 0$. For, we have shown in Theorem 41.2 using Mayer-Vietoris that

$$H^n_{dR}(S^n) = H^n_c(S^n) = \mathbb{R}.$$

  The integral of any positive form is positive, so $I_{S^n}$ is surjective; therefore its kernel is exactly the exact forms.

- $M = \mathbb{R}$. We have $H^0_c(\mathbb{R}) = 0$, since the only constant, compactly-supported function on $\mathbb{R}$ is zero. Meanwhile, we have $H^1_c(\mathbb{R}) = \mathbb{R}$, since a compactly-supported 1-form $\alpha$ on $\mathbb{R}$ is the derivative of a *compactly-supported* function if and only if $I_{\mathbb{R}}(\alpha) = 0$; indeed, the primitive $f(x) = \int_{-\infty}^x \alpha$ will have compact support if and only if $\int_{-\infty}^\infty \alpha = 0$.

  Notice that the answers for compactly-supported cohomology of $\mathbb{R}$ are reversed from the answers for ordinary cohomology; we will see that this is also true of $\mathbb{R}^n$, and reflects a general *duality* phenomenon that holds for all orientable manifolds.

## 48. Top cohomology (Mon 12/9)

Notice that in both cases of Example 47.2, we had $H^n_c(M) = \mathbb{R}$ and the integration map was an isomorphism. We will now prove that this is true of oriented manifolds in general. The proof will follow by induction from the case of $\mathbb{R}^n$ :

**Theorem 48.1.** *We have*

$$H_c^p(\mathbb{R}^n) = \begin{cases} 0 & 0 \le p < n \\ \mathbb{R} & p = n, \end{cases}$$

*and the integration map $I_{\mathbb{R}^n}$ is an isomorphism.*

The case $n = 1$ is the second bullet in Example 47.2, and the general case follows directly from:

**Lemma 48.2** (Compactly-supported Poincaré Lemma)**.** *Let $\omega \in \Omega_c^p(\mathbb{R}^n)$ be closed. If $1 \le p < n$, then there exists a compactly-supported $(p-1)$-form $\eta$ with $d\eta = \omega$. If $p = n$, there exists such an $\eta$ if and only if $\int_{\mathbb{R}^n} \omega = 0$.*

*Proof.* We have done the $n = 1$ case above, so assume $n \ge 2$.

Fix a large ball $B$ such that $\operatorname{supp} \eta \subset B$, and also take $B' \Supset B$.

By the ordinary Poincaré Lemma, there exists $\eta_0 \in \Omega^{p-1}(\mathbb{R}^n)$ with $d\eta_0 = \omega$; $\eta_0$ is not necessarily compactly supported, but we do have $d\eta = 0$ on $\mathbb{R}^n \setminus \bar{B}$. We now try to adjust $\eta_0$ to have compact support.

*Case 1.* $p = 1$. We have $\mathbb{R}^n \setminus \bar{B}$ is connected, so $\eta_0 = c$ is constant there. We may take

$$\eta = \eta_0 - c.$$

Then $\eta$ is compactly supported and $d\eta = d\eta_0 = \omega$.

*Case 2.* $1 < p < n$. As observed above, $\eta_0|_{\mathbb{R}^n \setminus \bar{B}}$ is closed. Let $S = \partial \bar{B}' \cong S^{n-1}$ be the boundary of $\bar{B}'$, which is a large sphere. We have $H^{p-1}(\mathbb{R}^n \setminus \bar{B}) \cong H^{p-1}(S) = 0$, since $p - 1 < n - 1$. Hence, there exists $\eta \in \Omega^{p-2}(\mathbb{R}^n \setminus \bar{B})$ such that $d\gamma = \eta_0$ there. Let $\psi$ be a bump function for $\mathbb{R}^n \setminus \bar{B}' \subset \mathbb{R}^n \setminus B$, and let

$$\eta = \eta_0 - d(\psi\gamma).$$

This vanishes identically on $\mathbb{R}^n \setminus \bar{B}'$, and satisfies $d\eta = d\eta_0 = \omega$ as desired.

*Case 3.* $p = n$ and $I_{\mathbb{R}^n}(\omega) = 0$. We have $H_{dR}^{n-1}(\mathbb{R}^n \setminus \bar{B}) \cong H_{dR}^{n-1}(S)$, where the isomorphism is induced by the restriction map. Hence, $\eta_0$ is exact if and only if its restriction to $S$ is exact; by the first bullet in Example 47.2, this is true if and only if $I_S(\eta_0|_S) = 0$. Applying Stokes's Theorem together with our assumption on $\omega$, we have

$$0 = \int_{\mathbb{R}^n} \omega = \int_{\bar{B}'} \omega = \int_{\bar{B}'} d\eta_0 = \int_S \eta_0.$$

Therefore $I_S(\eta_0) = 0$, so $\eta_0$ is exact on $S$, hence exact on $\mathbb{R}^n \setminus \bar{B}$, i.e. $\eta_0 = d\gamma$ there. As before, we can take $\eta = \eta_0 - d(\psi\gamma)$. $\square$

**Theorem 48.3.** *Let $M$ be a connected, oriented, smooth $n$-manifold. The integration map*

$$I_M : H_c^n(M) \xrightarrow{\sim} \mathbb{R}$$

*is an isomorphism. In particular, $H_c^n(M)$ is 1-dimensional.*

*Proof.* We can assume $n \ge 1$, since the result is obvious for a point. We already know that $I_M$ is surjective, since the integral of any positive form is positive. Hence, it suffices to prove that $I_M$ is injective: i.e., for $\omega \in \Omega_c^n(M)$, if $\int_M \omega = 0$ then there exists $\eta \in \Omega_c^{n-1}(M)$ such that $d\eta = \omega$.

We argue by induction. Let $\{U_i\}$ be a countable locally finite open cover with $U_i \cong \mathbb{R}^n$ for each $i$ (for instance, if $U_i$ are coordinate balls). Let

$$M_k = \cup_{i=1}^k U_i.$$

By relabeling the charts, we may assume that $M_k \cap U_{k+1} \neq 0$ for each $k$.

Since each compactly supported form is supported in $M_k$ for $k$ sufficiently large, it suffices to prove that for $\omega \in \Omega_c^n(M_k)$ with $\int_{M_k} \omega = 0$, there exists $\eta \in \Omega_c^{n-1}(M_k)$ such that $d\eta = \omega$. We will prove this statement by induction.

For the base case, we have $M_1 = U_1 \cong \mathbb{R}^n$, so the result follows immediately from Lemma 48.2.

Assume the result for $M_k$, and let $\omega \in \Omega_c^n(M_{k+1})$ with $\int_{M_{k+1}} \omega = 0$. Choose $\omega \in \Omega_c^n(M_k \cap U_{k+1})$ such that $\int_M \theta = 1$. Also let $\{\varphi, \psi\}$ be a partition of unity subordinate to $\{M_k, U_{k+1}\}$. Define

$$c = \int_{M_k} \varphi\omega.$$

Then $\operatorname{supp} \varphi\omega - c\theta \subset M_k$, and moreover

$$\int_{M_k} \varphi\omega - c\theta = 0$$

by choice of $c$. By the induction hypothesis, there exists $\alpha \in \Omega_c^n(M_k)$ such that

$$d\alpha = \varphi\omega - c\theta.$$

Next, observe that

$$0 = \int (\varphi + \psi)\omega = c + \int \psi\omega = \int (\psi\omega + c\theta).$$

Moreover, $\psi\omega + c\theta$ is supported in $U_{k+1} \cong \mathbb{R}^n$. By Lemma 48.2, there exists $\beta \in \Omega_c^n(U_{k+1})$ such that

$$d\beta = \psi\omega + c\theta.$$

Putting these together, we have

$$d(\alpha + \beta) = (\varphi\omega - c\theta) + (\psi\omega + c\theta) = (\varphi + \psi)\omega = \omega.$$

This completes the induction step, and the proof. □

**Corollary 48.4.** *For $M$ connected and orientable, we have:*

1. *If $M$ is compact then $H_{dR}^n(M) \cong \mathbb{R}$, with the integration map giving the isomorphism.*
2. *If $M$ is noncompact then $H_{dR}^n(M) = 0$.*

*If $M$ is connected and non-orientable, then $H_{dR}^n(M) = 0$.*

*Proof.* For (1), simply recall that $H_{dR}^n(M) = H_c^n(M)$ if $M$ is compact, and apply the previous Theorem.

For (2), one can construct a global primitive for any closed form using an exhaustion of $M$ by compact sets; see Lee, Theorem 17.32.

For the non-orientable case, one can pull back a top form to $\hat{M}$ to obtain a form with integral zero, construct a primitive there by Theorem 48.3, and take its average under the reversal map to obtain a primitive that descends back to $M$. See Lee, Theorem 17.34 for a different argument. □

## 49. The de Rham isomorphism (Mon 12/9)

We briefly recall the definition of singular homology and cohomology. Let

$$\Delta_p = \{x \in \mathbb{R}^{p+1} \mid \sum_{i=0}^{p} x_i = 1, 0 \leq x_i \leq 1\}$$

be the standard $p$-simplex. The space of **singular $p$-chains in $M$**

$$C_p(M, \mathbb{R})$$

is the free $\mathbb{R}$-vector space on the set of all maps $\Delta_p \to M$. A **chain** is a finite formal $\mathbb{R}$-linear combination of maps, $\sigma = \sum c_i \sigma_i$. For instance, $\Delta_p$ itself can be viewed as a chain in $C_p(\mathbb{R}^{p+1})$ via the inclusion map. The **boundary** $\partial \Delta_p \in C_{p-1}(\mathbb{R}^{p+1})$ is the formal sum of the faces of $\Delta_p$, with signs determined by the Stokes orientation. Extending the boundary operator $\partial$ linearly to chains, one can check that

$$\partial \circ \partial = 0.$$

Geometrically, this reflects the fact that the boundary of a manifold with boundary has no boundary.

Given a singular $p$-chain $\sigma \in C_p(M, \mathbb{R})$, we can form its boundary $\partial\sigma \in C_{p-1}(M, \mathbb{R})$ by restricting the values of each simplex to $\partial \Delta_n$. This operator again satisfies $\partial^2 = 0$, so we obtain a complex

$$\cdots \to C_{p+1}(M, \mathbb{R}) \xrightarrow{\partial} C_p(M, \mathbb{R}) \xrightarrow{\partial} \cdots.$$

The **singular homology** groups $H_p(M, \mathbb{R})$ are, by definition, the homology groups of this complex. Letting $C^p(M, \mathbb{R}) \coloneqq C_p(M, \mathbb{R})^*$ be the space of **singular $p$-cochains**, we can also form the dual complex

$$\cdots \xleftarrow{\delta} C^{p+1}(M, \mathbb{R}) \xleftarrow{\delta} C^p(M, \mathbb{R}) \xleftarrow{\delta} \cdots,$$

where $\delta$ is the transpose of $\partial$. The **singular cohomology** groups $H^p(M, \mathbb{R})$ are the homology groups of this complex.

Note that any linear functional on chains that vanishes on boundaries defines a cohomology class. (In fact, there is a natural map $H^p(M, \mathbb{R}) \to H_p(M, \mathbb{R})^*$ given by evaluating on a chain representing a homology class, and this map turns out to be an isomorphism since we are taking real coefficients.) A somewhat long but routine argument involving the Whitney Approximation Theorem, see Lee's book, shows that it is sufficient to define a linear functional on *smooth* chains, i.e. formal sums of smooth maps $\Delta_p \to M$, that vanishes on chains which are *smooth* boundaries. (In fact, the cohomology groups obtained from these so-called "smooth singular" cochains are isomorphic to the singular homology groups.) Integration against a *closed* form gives just such a functional, because of

**Theorem 49.1** (Stokes's Theorem, fourth version)**.** *Let $\sigma$ be a smooth singular $p$-chain in $M$ and $\omega \in \Omega^{p-1}(M)$. We have*

$$\int_{\partial\sigma} \omega = \int_{\sigma} d\omega.$$

Here, the integral over a chain $\sigma = \sum c_i \sigma_i$ is defined by

$$\int_{\sigma} \omega = \sum_i c_i \int_{\Delta_p} \sigma_i^* \omega.$$

In this way, we obtain the **de Rham map**

$$\begin{aligned} \mathscr{I}_p : H^p_{dR}(M) &\to H^p(M, \mathbb{R}) \\ [\omega] &\mapsto \int_* \omega. \end{aligned}$$

**Theorem 49.2.** *For any smooth manifold, the de Rham map is an isomorphism.*

The proof, see Lee Ch. 17, is a somewhat-more-involved version of the proof of Theorem 48.3 above. Notice that, combined with a calculation of the top singular cohomology of a manifold, as in Hatcher, the $p = n$ case follows from Corollary 48.4. Also, since

$$H^1(M, \mathbb{R}) = H_1(M, \mathbb{R})^* = (\pi_1(M, p)^{ab})^* = \operatorname{Hom}(\pi_1(M, p), \mathbb{R})$$

by the Hurewicz Theorem, the $p = 1$ case is Theorem 40.8.

## 50. Mapping degree (Wed 12/11)

We shall now discuss two applications of de Rham cohomology and in particular of Theorem 48.3. The first is to give a handy definition and interpretation of the *degree* of a map between compact oriented manifolds of the same dimension.

Recall that if $F : M \xrightarrow{\sim} N$ is a diffeomorphism and $\omega \in \Omega^n_c(N)$ is a top form, then we have

$$\int_M F^*\omega = \begin{cases} \int_N \omega & F \text{ orientation-preserving} \\ -\int_N \omega & F \text{ orientation-reversing.} \end{cases}$$

The first case is part of Definition/Lemma 45.1, and the second follows from the first because reversing orientation negates the integral. In the case that $M$ and $N$ are compact, we can generalize this result to arbitrary smooth maps, as follows.

**Theorem 50.1** (Lee, Theorem 17.35)**.** *Suppose that $M$ and $N$ are compact, oriented, $n$-dimensional smooth manifolds, and let $F : M \to N$ be a smooth map. There exists a unique integer $k \in \mathbb{Z}$, called the* **degree**, $\deg(F)$, *of $F$, with the following two properties:*

*(a) For any smooth $n$-form $\omega \in \Omega^n(N)$, we have*

$$\int_M F^*\omega = k \int_N \omega.$$

*(b) If $q \in N$ is a regular value of $F$, then*

$$k = \sum_{x \in F^{-1}(q)} \operatorname{sgn}(x),$$

*where*

$$\operatorname{sgn}(x) = \begin{cases} 1 & \textit{if } dF_x \textit{ is orientation-preserving} \\ -1 & \textit{if } dF_x \textit{ is orientation-reversing.} \end{cases}$$

**Remark 50.2.**
- Observe that the sum in (b) is finite: since $q$ is a regular value, by the inverse function theorem, $F^{-1}(q) \subset M$ is discrete, hence finite since $M$ is compact.
- Since $\omega$ is a top form, it is closed. By homotopy-invariance of the integral of the pullback of a closed form, (a) implies that if $F \simeq G$ are homotopic then $\deg(F) = \deg(G)$. Degree is therefore a homotopy invariant. In the case $N = S^n$, there is also a converse due to Hopf: two maps from a compact $n$-manifold to $S^n$ are homotopic if and only if they have the same degree. See Milnor, *Topology from the differentiable viewpoint*.

*Proof of Theorem 50.1.* By Theorem 48.3, we know that

$$[\omega] = [\omega'] \in H^n(N) \iff \int_N \omega = \int_N \omega'.$$

Let $\theta$ be any smooth $n$-form on $N$ such that $\int_N \theta = 1$. Define

$$k := \int_M F^*\theta.$$

(We do not yet know that $k \in \mathbb{Z}$.) To prove (a), let $\omega \in \Omega^n(N)$ and put $a = \int_M \omega$. We have $[\omega] = a\,[\theta]$, since these have the same integral. Since the pullback of cohomologous forms are cohomologous, we have

$$\int_M F^*\omega = a \int_M F^*\omega = ak = k \int_N \omega.$$

This proves (a).

Next, we prove (b). Suppose that $q$ is a regular value.

*Case 1.* $F^{-1}(q) = \{x_1, \dots, x_m\}$ is nonempty. Applying the inverse function theorem, we may choose a connected neighborhood $U \ni q$ and neighborhoods $V_i \ni x_i$ for $i = 1, \dots, m$, such that the restriction $F|_{V_i} : V_i \xrightarrow{\sim} U$ is a diffeomorphism for each $i$. Choose $\omega \in \Omega^n_c(U)$ such that $\int_U \omega = 1$. By (a), we have

$$k = \int_M F^*\omega = \sum_i \int_{V_i} F^*\omega = \sum \operatorname{sgn}(x),$$

as claimed.

*Case 2.* $F^{-1}(q) = \varnothing$. By compactness, $F(M) \subset N$ is closed. We can take $\omega \in \Omega^n_c(N \setminus F(M))$ with $\int_N \omega = 1$. But then $F^*\omega = 0$, so

$$\int_M F^*\omega = \int_M 0 = 0 \cdot \int_N \omega = k.$$

So both sides of (b) are zero, and we are done.

It remains to show that $k$ is in fact an integer. By Sard's theorem, the set of regular values of $F$ is nonempty. Letting $q$ be any regular value and applying (b), we learn that $k$ is an integer (since the sum on the RHS of (b) is manifestly so). ∎

We have the following generalization of Corollary 46.5.

**Theorem 50.3.** *Suppose $N$ is a compact, connected, oriented, smooth $n$-manifold and $X$ is a compact oriented smooth $(n+1)$-manifold with connected boundary. If a smooth map $f : \partial X \to N$ extends smoothly to $F : X \to N$, then* $\deg(f) = 0$.

*Proof.* Let $\omega \in \Omega^n(N)$ with $\int_N \omega = 1$. Using Stokes, we have

$$\deg(F) = \int_{\partial X} f^*\omega = \int_{\partial X} F^*\omega = \int_X dF^*\omega = \int_X F^*(d\omega) = 0,$$

since $d\omega \in \Omega^{n+1}(N) = 0$. □

**Corollary 50.4** (Brouwer Fixed-Point Theorem)**.** *Every (smooth) map from $\bar{B}^n$ to itself has a fixed point.*

*Proof.* Suppose, for contradiction, that $F : \bar{B}^n \to \bar{B}^n$ has no fixed points. Then we can define a smooth map $G : \bar{B}^n \to S^{n-1}$ by

$$G(x) = \frac{x - F(x)}{|x - F(x)|}.$$

Let

$$g = G|_{S^{n-1}} : S^{n-1} \to S^{n-1}.$$

By the previous Theorem, we have $\deg(g) = 0$. But we can also define

$$H : S^{n-1} \times I \to S^{n-1}$$
$$(x,t) \mapsto \frac{x - tF(x)}{|x - tF(x)|}.$$

(Note that the denominator here also does not vanish, since $x \neq F(x)$ and both belong to $S^n$.) This is a homotopy between the identity and $g$, which implies that $\deg(g) = \deg(\mathbf{1}) = 1$. We have reached a contradiction. □

## 51. Poincaré duality (Wed 12/11)

As you may know from algebraic topology, there is a *ring* structure on the cohomology groups of any space. This is particularly easy to describe in terms of de Rham cohomology. Suppose that $\omega$ and $\mu$ are closed forms of degrees $k$ and $\ell$, respectively, on a smooth manifold $M$. Then $\omega \wedge \mu$ is closed, by the Leibniz rule. Moreover, if $\mu = d\eta$ is exact, then

$$\omega \wedge \mu = (-1)^k d(\omega \wedge \eta)$$

is also exact. This shows that the wedge product descends to a map on cohomology classes,

$$\cup : H^k(M) \otimes H^\ell(M) \to H^{k+\ell}(M)$$
$$([\omega], [\mu]) \mapsto [\omega \wedge \mu] =: [\omega] \cup [\mu],$$

called the *cup product*. Notice that $[\omega] \cup [\mu] = (-1)^{k\ell} [\mu] \cup [\omega]$, so this makes $H^*(M)$ into a *graded-commutative* ring. It is clear from the definition that any smooth map $F : M \to N$ induces a *ring homomorphism*

$$F^* : H^*(N) \to H^*(M).$$

**Examples 51.1.** The following examples are well-known:

- $H^*(\mathbb{CP}^n) = \mathbb{R}[\alpha]/(\alpha^{n+1})$, $\alpha \in H^2(\mathbb{CP}^n)$.
- $H^*(\mathbb{T}^n) = \Lambda^*(\mathbb{R}^n)$, where $\mathbb{R}^n = \langle [dx^1], \dots, [dx^n] \rangle$.

Suppose further that $\mu$ is compactly supported; then $\omega \wedge \mu$ is again compactly supported. So the cup product induces a map

$$H^k(M) \otimes H^\ell_c(M) \to H^{k+\ell}_c(M).$$

Now assume that $M$ is oriented and take $\ell = n - k$. We can post-compose with the integration map to obtain a pairing

$$H^k(M) \otimes H^{n-k}_c(M) \overset{\cup}{\to} H^n_c(M) \overset{I_M}{\to} \mathbb{R}. \tag{51.1}$$

We have the following fundamental fact about orientable manifolds.

**Theorem 51.2** (Poincaré duality). *The pairing (51.1) is* perfect*, i.e., induces isomorphisms*

$$H^{n-k}_c(M) \overset{\sim}{\to} H^k(M)^*$$
$$H^k(M) \overset{\sim}{\to} H^{n-k}_c(M)^*.$$

The theorem can be proved by imitating the proof of the de Rham theorem in Lee's book, which in turn is a version of the proof of Theorem 48.3 above. You will do so on homework. Another, perhaps more satisfying, proof can be given using *Hodge theory*, which we hope to develop in Math 765.